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ON THE CONVERGENCE AND SATURATION PROBLEM OF A CLASS OF DISCRETE LINEAR OPERATORS OF ENTIRE EXPONENTIAL TYPE IN $L_p(-\infty,\infty)$ SPACES

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We consider the following discrete, linear operators

$$J_\sigma(f,x)=\sum_{k=-\infty}^{\infty} f(t_k)\,\frac{\sin^2\frac{\sigma}{2}(x-t_k)}{(\frac{\sigma}{2}(x-t_k))^2},\quad t_k=\frac{2k\pi}{\sigma},\quad k=0,\pm1,\pm2,\pm3,\ \dots\ ,$$

where $f(x)$ is a fixed bounded function in $L_p(-\infty,\infty)$.

The first saturation result of this type operators(trigonometric case) is obtained by J.Szabados [1] ($p=\infty$) and by V.Popov-J.Szabados [2] ($1\leq p<\infty$).The operators of the analogous type are considered by Butzer, Splettstößer and Stens[3] in a connection with the Shannon theorem.The convergence in C-metric of $J_\sigma(f,x)$ is due to Schmeisser.

The purpose of this paper is to give different types of estimates for $\|J_\sigma(f,x)-f(x)\|_{L_p}$, $p\geq 1$ and to solve the saturation problem for this approximation process.

Let's introduse the following notations:

$$F_\sigma(x)=\frac{\sigma}{2\pi}\,\frac{\sin^2\frac{\sigma x}{2}}{(\frac{\sigma x}{2})^2}\quad\text{- the kernel of Fejer's type,}$$

$B_{\sigma,p}$ - the set of the entire functions of an exponential type $\sigma>0$ which belong to the space $L_p(-\infty,\infty)$, The following norms will be employed

$$\|g\|_{L_p'} = \begin{cases} \text{ess sup}(\,|g(x)|\,,\ x \in (-\infty,\infty)), & p=\infty \\ \|g\|_{L_p}\,, & 1 \leq p < \infty\ ; \end{cases}$$

$$\|g\|_{L_\infty} = \sup(\,|g(x)|, x \in (-\infty,\infty)\,)$$

The operators

$$J_\sigma(f,x) = \frac{2\pi}{\sigma} \sum_{k=-\infty}^{\infty} f(t_k) F_\sigma(x - t_k)$$

posses the following interpolation properties

$$J_\sigma(f,t_k) = f(t_k)\ ,\ J_\sigma'(f,t_k) = 0\ ,\ J_\sigma(1,x) = 1\ .$$

If $\|f\|_{L_\infty} = M$, then $J_\sigma(f,x) \in B_{\sigma,\infty}$ and $\|J_\sigma(f,x)\|_{L_\infty} \leqq M$.

Our first result give an error estimate in terms of the modulus of continuity in L_∞. Applying the inequality

$$F_\sigma(x) \leqq \min\left(\frac{2}{\pi\sigma x^2}, \frac{\sigma}{2\pi}\right)$$

we obtain the following theorem-

Theorem 1. If $f(x) \in L_\infty$, then the following estimates hold:

a) $$\| f(x) - J_\sigma(f,x) \|_{L_\infty} = O(1) \sum_{i=1}^{\infty} i^{-2}\, \omega(f, \tfrac{i}{\sigma})_{L_\infty}\ ,$$

b) $$\| f(x) - J_\sigma(f,x) \|_{L_\infty} = O(1)\left(\omega(f, \frac{\ln\sigma}{\sigma})_{L_\infty} + \frac{\|f\|_{L_\infty}}{\ln\sigma}\right) -$$

if $\omega(f,h)_{L_\infty} \xrightarrow[h\to 0]{} 0$, then $\| f(x) - J_\sigma(f,x) \|_{L_\infty} \xrightarrow[\sigma\to\infty]{} 0$,

c) if $f(x) \in \mathrm{Lip}\,1$, then $\| f(x)-J\ (f,x)\|_{L_\infty} = O(\frac{\ln\sigma}{\sigma})$, $\sigma \longrightarrow \infty$,

i.e. the optimal expected order isn't attained.

The following theorem is obtained in terms of the averaged modulus of smoothness- $\tau(f,h)_{L_1}$ which is due to Sendov-Korovkin [4]

Theorem 2. If $f(x) \in L_1 \cap L_\infty$, then the following estimates hold

a) $$\|f(x)-J_\sigma(f,x)\|_{L_1} = O(1)\Big(\sum_{i=1}^{[\sigma]} i^{-2}\tau(f,\frac{i}{\sigma})_{L_1} + \frac{\|f\|_{L_1}}{\sigma}\Big)$$

b) from a) we obtain ($\sigma \geqq 1$)

$$\|f(x)-J_\sigma(f,x)\|_{L_1} = O(1)\ \Big(\frac{1}{\sigma}\int_{1/2}^{\sigma}\tau(f,\frac{1}{u})_{L_1}du + \tau(f,\frac{1}{\sigma})_{L_1} + \frac{\|f\|_{L_1}}{\sigma}\Big)$$

if $\tau(f,h)_{L_1} \xrightarrow[h\to 0]{} 0$, then $\|f(x)-J_\sigma(f,x)\|_{L_1} \longrightarrow 0$, $\sigma \longrightarrow \infty$,

c) if $\tau(f,h)_{L_1} = O(h)$, then $\|f(x)-J_\sigma(f,x)\|_{L_1} = O(\frac{\ln\sigma}{\sigma})$.

The estimates which we have obtained don't give the expected optimal order $O(\frac{1}{\sigma})$. In order to get a better estimate we prove the following theorem 3 which also settles the case $1<p<\infty$.

Theorem 3. If $f(x) \in L_p \cap L_\infty$, then

$$\|f(x)-J_\sigma(f,x)\|_{L_p} = \begin{cases} O(1)\big(\tau(f,\frac{1}{\sigma})_{L_1} + \omega(\tilde{f},\frac{1}{\sigma})_{L_1}\big), & p=1, \\ O(1)\ \tau(f,\frac{1}{\sigma})_{L_p}, & 1<p<\infty, \\ O(1)\big(\tau(f,\frac{1}{\sigma})_{L_\infty} + \omega(\tilde{f},\frac{1}{\sigma})_{L_\infty}\big), & p=\infty, \end{cases}$$

where $\tilde{f}(x)$ is the Hilbert transform of $f(x)$.

In the case $1<p<\infty$ we use the M. Riesz inequality. We note, that

$$\tau(f,\frac{1}{\sigma})_{L_\infty} = \omega(f,\frac{1}{\sigma})_{L_\infty}\ .$$

First we define the Hilbert transform on the real line for a subspace of L'_∞ following Achieser [5] and Koizumi [6]. We have:

Definition (Achieser-Koizumi). If $f'(x) \in L_\infty$, then

$$\tilde{f}'(x) = -(x-i)\,\overbrace{i\,\mathrm{sign}(*)\widehat{\frac{f'(\cdot)}{(\cdot)-i}}(*)}^{\vee}(x) = \frac{x-i}{\pi}\int_{-\infty}^{\infty}\frac{f'(t)}{t-i}\,\frac{dt}{x-t},\quad i^2=-1,$$

where $\wedge$ is the Fourier transform and $\vee$ is the converse Fourier transform.

We use the following two definitions:

Definition. If $f(x) \in W^1_\infty$, then

$$\tilde{f}'(x) \overset{\text{def.}}{=} -\overbrace{i\,\mathrm{sign}(*)\widehat{\frac{f'(\cdot)}{(\cdot)-i}}(*)}^{\vee}(x)\,(x-i) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t-i)^2}dt\ .$$

Definition. If $g(x) \in L_\infty$, then $g(x+h)-g(x) \overset{\text{def.}}{=} \left(\widetilde{\int_{\cdot}^{\cdot+h} g(t)dt}\right)'(x)$.

The proof of th 3 consists of a series of 5 lemmas each one of them being of particular interest. We denote:

$$\Omega_\sigma(f,x) = \int_{-\infty}^{\infty} f(x+t)F_\sigma(t)dt,\quad F_\sigma(t) = \frac{\sigma}{2\pi}\,\frac{\sin^2\frac{\sigma t}{2}}{\left(\frac{\sigma t}{2}\right)^2},\ \tilde{F}_\sigma(t) = \frac{1}{\pi\sigma}\,\frac{\sigma t-\sin\sigma t}{t^2}.$$

Lemma 1. If $f(x) \in W^1_p$, $\tilde{f}'(x) \in L'_p$, then

$$\left\|\Omega_\sigma(f,x) - f(x)\right\|_{L_p} = O(1)\left\|\tilde{f}'\right\|_{L'_p}\ .$$

The second lemma is concerned with the Jackson means:

$$Q_\sigma(f,x) = c_\sigma\int_{-\infty}^{\infty} f(x+t)F^2_\sigma(t)dt,\ Q_\sigma(1,x) \equiv 1 \implies c_\sigma = O\left(\frac{1}{\sigma}\right)\ .$$

Lemma 2. If $f(x) \in W^1_p$, then

a) $\left\|f(x) - Q_\sigma(f,x)\right\|_{L_p} = O\left(\frac{1}{\sigma}\right)\left\|f'\right\|_{L'_p}$,

b) $\left\| Q'_\sigma(f,x) \right\|_{L_p} = O(1) \left\| f' \right\|_{L'_p}$.

Lemma 3 is an application of the Riesz-Thorin interpolation theorem:

Lemma 3. If $f(x) \in W_p^1$,then

$$\left(\frac{1}{\sigma} \sum_{k=-\infty}^{\infty} \left| f(t_k) - Q_\sigma(f,t_k) \right|^p \right)^{\frac{1}{p}} = O\left(\frac{1}{\sigma}\right) \left\| f' \right\|_{L_p} .$$

Lemma 4.Let f(x) be a given function.Then

$$\left\| J_\sigma(f,x) \right\|_{L_p} = O(1) \left(\frac{1}{\sigma} \sum_{k=-\infty}^{\infty} \left| f(t_k) \right|^p \right)^{\frac{1}{p}} .$$

Lemma 5. If $t_\sigma(x) \in B_{\sigma,p}$, then

$$\left\| J_\sigma(t_\sigma,x) - \Omega_\sigma(t_\sigma,x) \right\|_{L_p} = O\left(\frac{1}{\sigma}\right) \left(\left\| t'_\sigma(x) \right\|_{L_p} + \left\| \tilde{t}'_\sigma(x) \right\|_{L_p} \right) .$$

In the proof of theorem 3 we use as an intermediate approximation tool the Steklov transform

$$f_\sigma(x) = \frac{\sigma}{2} \int_{-\frac{1}{\sigma}}^{\frac{1}{\sigma}} f(x+t)dt .$$

We have

$$\left\| f(x) - J_\sigma(f,x) \right\|_{L_p} = \left\| f(x) - f_\sigma(x) \right\|_{L_p} + \left\| f_\sigma(x) - \Omega_\sigma(f_\sigma,x) \right\|_{L_p} +$$

$$\left\| \Omega_\sigma(f_\sigma - Q_\sigma(f_\sigma),x) \right\|_{L_p} + \left\| \Omega_\sigma(Q_\sigma(f_\sigma),x) - J_\sigma(Q_\sigma(f_\sigma),x) \right\|_{L_p}$$

$$+ \left\| J_\sigma(Q_\sigma(f_\sigma) - f_\sigma,x) \right\|_{L_p} + \left\| J_\sigma(f_\sigma - f,x) \right\|_{L_p}$$

and using the lemmas 1-5 we can obtain the proof of th 3.

Now we turn to the saturation problem of $J_\sigma(f,x)$.This problem is solved in the space

$$X_p = \left[f(x) \in L_p \cap L_\infty , \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x+t)dt \xrightarrow[\varepsilon \to 0]{} f(x) \right]$$

The following theorem is proved:

Theorem 4. If $f(x) \in X_p$, then

$$1) \quad \left\| f(x) - J_\sigma(f,x) \right\|_{L_p} = O\left(\frac{1}{\sigma}\right)$$

if and only if

a) $p=1$, both $f(x)$ and $\tilde{f}(x)$ are of bounded variation,

b) $p \in (1,\infty)$, $f(x) \in W_p^1$,

c) $p=\infty$, both $f(x)$ and $\tilde{f}(x)$ belong to the class Lip 1;

$$2) \quad \left\| f(x) - J_\sigma(f,x) \right\|_{L_p} = \bar{o}\left(\frac{1}{\sigma}\right) \Longleftrightarrow f(x) \equiv \text{const}.$$

The proof of a th 4 is based on the th 3 and on the following equalities

$$1. \quad \frac{\sin\sigma x}{\sigma} J'_\sigma(f,x) = J_{2\sigma}(J_\sigma(f),x) - J_\sigma(f,x) .$$

$$2. \quad \sin\sigma x \, J'_\sigma(f,x) = (1-\cos\sigma x)\, \tilde{J}'_\sigma(f,x) .$$

Assuming that

$$\left\| J_\sigma(f,x) - f(x) \right\|_{L_p} = O\left(\frac{1}{\sigma}\right)$$

we get the following estimate

$$\left\| J'_\sigma(f,x) \right\|_{L_p} = O(1)$$

which play the basic role in the proof.

Note. If $f(x)$ is 2π - periodic , bounded function, then we have

$$J_\sigma(f,x) = \frac{1}{n^2} \sum_{k=0}^{n-1} f(q_k) \frac{\sin^2 \frac{n}{2}(x-q_k)}{\sin^2 \frac{x-q_k}{2}}, \quad q_k = \frac{2k\pi}{n}, \; k=0,1,\dots,n-1;$$

This operator is considered by Szabados and Popov [1] , [2] .Theorem 4,1),c) and 2) gives the saturation result from [1] .

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