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DIFFICULTIES IN RATIONAL CHEBYSHEV APPROXIMATION

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Let X be a compact space and $\|\ \ \|$ denote the Chebyshev (maximum, uniform) norm on $C(X)$. Consider approximation of continuous f by a fixed family of (generalized) rational functions of the form $R(A,.) = P(A,.)/Q(A,.)$ where $Q(A,x) > 0$ for $x \in X$. This problem is stated and studied in the text of Cheney (1966, Chapter 5). There are a large number of known analytical and numerical difficulties in approximation by such families. In this note we point out some new ones.

1. Difficulties with Non-existence

Define $\rho(f) = \inf\{\|f-R(A,.)\| : Q(A,.) > 0\}$. We consider the case in which no best approximation exists, that is, no A exists for which the infimum is attained. In principle non-existence might be thought of no practical significance since

(i) by definition of ρ, there exists a sequence $\{A^k\}$ of coefficient vectors such that $Q(A^k,.) > 0$ and (*) $\|f-R(A^k,.)\| \to \rho(f)$

(ii) for practical applications all we need is A such that $Q(A,.) > 0$ and $\|f-R(A,.)\|$ is slightly larger than $\rho(f)$

(iii) there exist algorithms (for example, differential correction (Cheney, 1966, p. 171; Lee and Roberts, 1973) and the linear inequality method (Cheney, 1966, p. 170; Lee and Roberts, 1973)) such that (in theory at least) the sequence of coefficient vectors produced satisfies (*)

(iv) due to truncation and rounding error, no algorithm can be relied on to produce an exact best approximation.

Points (i-iv) would seem to suggest that we merely take $R(A^k,.)$ for k sufficiently large. Unfortunately, we must point out an unrecognized difficulty with $R(A^k,.)$ for all k large. Let us normalize rationals so that the sum of absolute values of denominator coefficients is 1. By standard compactness arguments (Dunham, 1977, p. 285), $\{A^k\}$ is bounded and has an accumulation point A. By taking a subsequence if necessary, we can assume $\{A^k\} \to A$. As $Q(A^k,.) > 0$, we have $Q(A,.) \geq 0$. As $\|f-R(A,.)\| \leq \rho(f)$ (by standard arguments using Goldstein's convention (Rice, 1969, p. 85ff)), $Q(A,.)$ must have a zero at some point z of X, hence

$P(A,.)$ must have a zero at the same point z . It follows that for given $\varepsilon > 0$, $|P(A^k,z)|$ and $Q(A^k,z)$ are less than ε for all k sufficiently large . This suggests that evaluation of $P(A^k,.)$ and $Q(A^k,.)$ will be unstable near z (in the case of ordinary rational approximation, $x-z$ will be nearly a common factor of $P(A^k,.)$ and $Q(A^k,.)$). Instability of evaluation of $P(A^k,.)$ and $Q(A^k,.)$ will cause trouble with algorithms which purport to do (*).

A further difficulty with cases of non-existence is that the differential correction algorithms are unstable for such cases. For discussion of DC and notation, see Dunham (1980).

EXAMPLE: Let $X = \{0,1\}$ and f take the values $\{1,0\}$. Approximate by ratios of constants to first degree polynomials. f is a limit of approximations so $\rho(f) = 0$. Let $\Delta = \Delta^* = \rho(f) = 0$. Plug $P(A,x) / Q(A,x) = \gamma/x$ into (1) of Dunham (1980) to get

$$\delta(A) = \max \left\{ \frac{|\gamma|}{w(0)} , \frac{|\gamma|}{w(1)} \right\}$$

For γ small, $\delta(A)$ is small, but $R(A,.)$ has a pole at 0. Since δ depends continuously on Δ for Δ close to the optimum, A will give a low value for Δ and γ small.

Consider approximation by generalized rational functions. There exists C such that $\|f-R(C,.)\| \le \rho(f)$, $Q(C,.) \ge 0$ and $P(C,.),Q(C,.)$ have a common zero at x_0 by Goldstein's theory (Rice, 1969, p. 85ff). Let $\Delta = \Delta^* = \rho(f)$ and for γ small, $P(A,.)/Q(A,.) = (P(C,.) + \gamma)/Q(C,.)$. Plugging A into (1) of Dunham (1980) gives

$$\delta(A) = \max \left\{ \frac{\left| \left(\frac{P(C,.)}{Q(C,.)} + \varepsilon\Delta^* \right) Q(C,.) - P(C,.) - \gamma \right| - \Delta^* Q(C,.)}{w} \right\}$$

$$= \max \left\{ \frac{| \varepsilon \Delta^* Q(C,.) - \gamma | - \Delta^* Q(C,.)}{w} \right\} \le \gamma/w(x) \quad ,$$

(where $Q(C,.)$ vanishes, $R(C,.)$ is multiplied by zero). But $R(A,.)$ has a non-removable pole at x_0 .

It should be noted that non-existence is not uncommon in discrete approximation of arbitrary functions. Consider for example approximation on a two-point set $X = \{x_1,x_2\}$ by $R_1^0(X)$. Non-existence occurs except in the cases $f \equiv 0$, $f(x_1) = -f(x_2)$, $f(x_1)f(x_2) > 0$. It would be valuable to have an analysis of the probability of non-existence in more complicated cases.

It would be worthwhile to have an <u>a posteriori</u> test for non-existence. We have seen that approximations tend to an approximation with at least one pole. Unfortunately, degenerate cases tend to exhibit this tendency also and near degenerate cases will likely produce approximations almost having a pole. It should be noted that degenerate cases can be perturbed into cases of non-existence Dunham (1982), suggesting that the asked for test of the previous paragraph may not be stable.

2. Non-existence of Pole-Free Approximations on Supersets

In this section we modify the problem slightly. The problem is, given a set W, a compact proper non-empty subset X of W, and a function f continuous on X, to maximize $\|f - p/q\|$ subject to the constraint $q > 0$ on W.

Some such problems arise in applications. Consider the problem of obtaining an approximation on a set W which is infinite or a very large finite set. As most simple algorithms (Cheney, 1966, p. 170ff; Lee and Roberts, 1973) for rational Chebyshev approximation work well only on small finite sets, we generally replace approximation on W by approximation on a (small) finite subset X of W. Secondly, we might be interested in approximations for the unbounded intervals $[0,\infty)$ or $(-\infty,\infty)$, but might decide to approximate on the bounded intervals $[0,k]$ or $[-k,k]$ respectively for k large as a substitute. Thirdly, we could have scientific data, with X the set of points on which observations were made and W the set to which the model applies. Fourthly, in the rational second algorithm of Remez for an interval $[\alpha,\beta]$, we seek a best approximation on a trial alternant (reference) X which is pole-free on $[\alpha,\beta]$. Fifthly, in the rational first algorithm of Remez, we desire approximations on the finite subsets to be pole-free on the full set.

Now if some best approximation p^*/q^* on X has a positive denominator on W, it can be used as an approximation on W. (REMARK: By Dunham (1979) a best approximation by ordinary rationals must be unique). We consider the case in which such p^*/q^* does not exist. We might then ask for p_0/q_0 which is best on X among rationals with denominator > 0 on W, a <u>non-standard</u> approximation problem (in the standard problem, in Cheney (1966, Chapter 5), denominators are required to be > 0 only on the domain of approximation). Using betweeness (Dunham, 1969), we could give a <u>formal</u> characterization of p_0/q_0. Also we might seek algorithms for computing p_0/q_0.

THEOREM: Let there exist no p^*/q^* best on X which happens to have positive denominator on compact W. Then p_0/q_0 cannot exist.

<u>Proof</u>: Suppose it did exist. Let p_1/q_1 be better on X than p_0/q_0 and define

$$r_\lambda = p_\lambda/q_\lambda = [(1-\lambda)p_0 + \lambda p_1]/[(1-\lambda)q_0 + \lambda q_1] .$$

As $\{q_\lambda\} \to q_0$ uniformly on compact W as $\lambda \to 0$, $q_\lambda > 0$ on W for all λ sufficiently small. By betweeness arguments of the author (1969, pp. 152-153), since p_0/q_0 is not best on X with denominators > 0 on X, r_λ is better on X for all λ sufficiently small. But r_λ has positive denominators on W for all λ sufficiently small, contradicting the supposed optimality of p_0/q_0.

There are unfortunately cases of interest where W is not compact, in particular the cases $W = [0,\infty)$ and $W = (-\infty,\infty)$ with approximation by ordinary rationals. Whether p_0/q_0 can exist in this case appears to be a difficult problem. We can, however, at least show that

THEOREM: Let X be compact. Let the highest denominator power exceed the highest numerator power. Let the coefficient of the highest denominator power in q_0 be non-zero. Then p_0/q_0 cannot be best on X under the constraint $q > 0$ on $W = [0,\infty)$ or $(-\infty,\infty)$.

Proof: Define r_λ as before. By (Dunham, 1981, p. 171) $\{q_\lambda\} \to q_0$ uniformly as $\lambda \to 0$, hence $q_\lambda > 0$ for λ sufficiently small. By previous arguments r_λ is better on compact X for λ small.

What happens in mean (L_p) approximation problems is open. The arguments used earlier do not fully apply, as there can exist locally best approximations which are not globally best.

3. Largeness of the Lipschitz Constant

In this section we consider general non-linear approximation (the result and in particular some references do apply to rational approximation). f and g (possibly with subscripts) denote functions being approximated. We consider only problems in which a best approximation (if it exists) is unique but do not assume global uniqueness. Denote a best approximation to h by Th. f is said to have a Lipschitz constant $L = L(f)$ if for all g

$$\|Tf - Tg\| \le L\|f-g\| \quad .$$

Sufficient conditions have been given by Cheney (1966, p. 82, p. 168), Maehly and Witzgall (1960, p. 300) and by Barrar and Loeb (1970) for f to have a Lipschitz constant. Conversely, if T is discontinuous at f, sufficient conditions for which have been given by Werner (1964) and Schmidt (joint with the author) (1979), there exists $\{f_k\} \to f$ with $\{Tf_k\} \not\to Tf$ and hence (for possibly a subsequence of $\{f_k\}$)

(1) $$\|Tf_k - Tf\| > k\|f-f_k\| \quad ,$$

hence no Lipschitz constant can hold for f. Observe that (1) implies a Lipschitz constant of greater than k for f_k. The following lemma was communicated by R. Bojanic in response to earlier work by the author.

LEMMA: Let there exist no Lipschitz constant for f. Let $g_k \to f$ and Tg_k exist. If $L(g_k)$ exists for every $k = 1,2,3,\ldots$ then $\limsup_{k\to\infty} L(g_k) = \infty$.

Proof: Since there exists no Lipschitz constant for f, there is a sequence (f_r) such that

$$\|T(f) - T(f_r)\| \geq r\|f-f_r\| \quad .$$

Since $g_k \to f$, there is a subsequence (g_{k_r}) such that

$$\| g_{k_r} - f \| \leq \frac{1}{2} \| f - f_r \| \ .$$

Since each g_k has a Lipschitz constant $L(g_k)$, we have

$$\begin{aligned} r \| f - f_r \| &\leq \| T(f) - T(f_r) \| \\ &\leq \| T(f) - T(g_{k_r}) \| + \| T(g_{k_r}) - T(f_r) \| \\ &\leq L(g_{k_r}) \left(\| f - g_{k_r} \| + \| g_{k_r} - f_r \| \right) \\ &\leq L(g_{k_r}) (2 \| f - g_{k_r} \| + \| f - f_r \|) \\ &\leq 2 L(g_{k_r}) \| f - f_r \| , \end{aligned}$$

or $L(g_{k_r}) \geq \frac{1}{2} r$, and the lemma is proved.

THEOREM: Under the same hypotheses, $\lim_{k \to \infty} L(g_k) = \infty$.

Proof: Suppose $\lim_{k \to \infty} \inf L(g_k) < \infty$, then we can extract a subsequence with it as limit and then apply the lemma to it, giving a contradiction.

The significance of the theorem is that whereas the cited results of Cheney, Maehly and Witzgall and Barrar and Loeb tend to suggest that g with a Lipschitz constant is well behaved, the theorem emphasizes that if g is near f with no Lipschitz constant, the Lipschitz constant of g could be large enough for bad behaviour.

4. Smallness of Constant of Strong Uniqueness

A sufficient condition for existence of a Lipschitz constant is strong uniqueness, used by by Cheney and by Barrar and Loeb.

THEOREM: Let f have zero as its largest strong uniqueness constant and $\{g_k\} \to f$. Then the sequence of strong uniqueness constants of $\{g_k\}$ has limit zero.

This is an immediate consequence of a result of Phelps proven by Bartelt (1975). The approach of Cheney and of Barrar and Loeb suggest that g with positive strong uniqueness constant is well behaved, whereas the above theorem suggests that if g is near f, the strong uniqueness constant of g may be so low as to be useless.

5. Small Denominators with Near Degeneracy

Consider approximation of continuous g. In this section it is suggested that the approximation problem is numerically ill-posed if g is close to a function f possessing a degenerate best approximation. For ordinary rational approximation on an interval degeneracy is the deficiency d of Rivlin (1969, p. 126), the quantity d in the text of Rice (1964, p. 78) and the quantity d in the text of Achieser (1956, p. 52). For generalized rationals degeneracy is the deficiency in the dimension of the associated linear space (Cheney, 1966, Chapter 5: Cheney and Loeb, 1966). Let Tg denote the best approximation to g and rationals have normalized denominators, such as the normalization (3-8.2) of the text of Rice (1964, p. 75) or of Dunham (1982, (1)).

THEOREM: Consider approximation on finite interval $[\alpha,\beta]$ by ordinary rationals $R_m^\ell[\alpha,\beta]$. Let Tf be degenerate and $f-Tf$ have fewer than $\ell+m+1$ alternations on $[\alpha,\beta]$. Let $\{f_k\} \to f$ and $Tf_k = R(A^k,.)$ be non-degenerate. Then
(2) $\inf\{Q(A^k,x) : \alpha \leq x \leq \beta\} \to 0$.

Proof: Suppose not, then by taking a subsequence if necessary we can assume

(3) $\inf\{Q(A^k,x) : \alpha \leq x \leq \beta\} > \varepsilon$.

We find it convenient to allow denominators $Q(A,.) \geq 0$. It is straightforward to show that this does not change solutions to the approximation problem (in this case) (Rice, 1964, Lemma 3-7; Dunham, 1983, 337). Now $\{A^k\}$ has an accumulation point A^0 and A^0 is best under the constraint $Q(A,.) \geq 0$ (Dunham, 1982). By taking a subsequence if necessary assume $\{A^k\} \to A^0$, hence $Q(A^0,.) \geq \varepsilon$. Then $R(A^k,.) \to R(A^0,.)$ uniformly on $[\alpha,\beta]$. But $f_k - R(A^k,.)$ has at least $\ell+m+2$ alternating extrema on $[\alpha,\beta]$, hence it cannot converge uniformly to $f - Tf$. We have a contradiction to (3).

It has been pointed out by Kaufman and Taylor (1981) that numerical algorithms have difficulty with small denominators and that approximations with small denominators may not be useful.

The results of Werner (1964) guarantee (2) for one fixed sequence $\{f_k\} \to f$ without alternation hypotheses on $f - Tf$.

THEOREM: Consider generalized rational approximation on a finite set X. Let the linear space associated with each rational be a Haar subspace on X. Let $\{f_k\} \to f$, $Tf_k = R(A^k,.)$ be non-degenerate, and $f - Tf$ have fewer extrema than non-degenerate cases require. Then $\inf\{Q(A^k,x) : x \in X\} \to 0$.

Proof: Argue similarly to the previous theorem. By arguments of Dunham (1968, p. 486), $R(A^k,.)$ is uniquely best under the constraint $Q(A,.) \geq 0$. $\{A^k\}$ has an accumulation point A^0 which is best under the same constraint by Dunham (1982) and by hypothesis $Q(A^0,.) \geq \varepsilon$. Assuming $\{A^k\} \to A$, we get $f-R(A^k,.) \to f-R(A^0,.)$ uniformly. But $f-R(A^0,.)$ has too few extrema and we have a contradiction.

6. Small Denominators with Near Non-Existence

As in the previous section, we normalize denominators.

THEOREM: Let Tf not exist. Let $\{f_k\} \to f$ and $Tf_k = R(A^k,.)$. Then $\inf\{Q(A^k,x) : x \in X\} \to 0$.

Proof: Suppose not, then by previous arguments we can assume $Q(A^k,.) > \varepsilon$ for all k large. $\{A^k\}$ has an accumulation point A^0 by previously cited arguments. By taking a subsequence if necessary we can assume $\{A^k\} \to A^0$. As $Q(A^0,.) \geq \varepsilon$, $R(A^k,.) \to R(A^0,.)$ uniformly. By the well-known continuity of $\rho(g)$ as a function of g, $R(A^0,.)$ must be best to f. This is a contradiction of non-existence, proving the theorem.

7. Non-Existence with Interpolation for $R^{\ell}_m[\alpha,\beta]$

In uniform approximation with Lagrange-type interpolation there is a set of nodes Z and we require $R(A,z) = f(z)$ for all $z \in Z$. A theory for such approximation by varisolvent families has been obtained by Barrar and Loeb (1969). An isolated example of non-existence by ordinary rational functions $R^{\ell}_m[\alpha,\beta]$ has been given by Loeb (1968). Degeneracy has been defined in Section 5.

THEOREM: Let the best approximation $r^* = p^*/q^*$ by ordinary rational functions (without interpolation) be degenerate. Let the set of nodes Z be the endpoint α and $f(\alpha) \neq r^*(\alpha)$. Then f has no best approximation with interpolation on Z.

Proof: For $r \in R^{\ell}_m[\alpha,\beta] \sim r^*$, $\|f-r^*\| < \|f-r\|$ by classical uniqueness results. Let p^*,q^* be relatively prime and define

$$r_k(x) = p^*(x)/q^*(x) + [f(\alpha) - r^*(\alpha)]/[1+k(x-\alpha)],$$

then $r_k(\alpha) = f(\alpha)$. Assume without loss of generality that $f(\alpha) - r^*(\alpha) > 0$. Let w be the smallest zero of $f-r^*$ on $(\alpha,\beta]$. As $f-r^*$ is ≥ 0 on $[\alpha,w]$ and r_k-r^* is positive and decreasing on $[\alpha,w]$, it is seen by drawing a diagram that $|f-r_k| \leq \|f-r^*\|$ on $[\alpha,w]$. As $\{r_k\} \to r^*$ uniformly on $[w,\beta]$,

$$\max\{|f(x) - r_k(x)| : w \leq x \leq \beta\} \to \|f-r^*\| .$$

Thus $\|f-r_k\| \to \|f-r^*\|$.

THEOREM: Let the best approximation $r^* = p^*/q^*$ by ordinary rational functions (without interpolation) have degeneracy ≥ 2. Let Z be a single internal node z and let $f(z) \neq r^*(z)$. Then f has no best approximation with interpolation on Z.

Proof: The arguments are similar with

$$r_k(x) = p^*(x)/q^*(x) + [f(z) - r^*(z)]/[1+k(x-z)^2] .$$

The author and D. Schmidt (Oakland University) have independently obtained non-existence examples in which the best (ordinary) approximation is non-degenerate.

Consider complex approximation of real f by complex ordinary rational functions on a real interval $[\alpha,\beta]$, studied by Saff and Varga (1978). In case f has a unique real degenerate best approximation, the theorems apply.

The arguments of the first theorem can be applied to approximation by other non-linear families. For example, if we approximate by exponentials (Werner, 1970) on $[0,\beta]$, we can define an approximant r to be degenerate if r plus any single exponential term is an element of the family, and choose

$$r_k(x) = r^*(x) + [f(\alpha) - r^*(\alpha)]\exp(-kx) \quad .$$

The situation for mean (L_p) approximation is also of interest. Unfortunately the theorems fail to apply for $1 < p < \infty$ since a degenerate approximation r^* cannot be best to $f \not\equiv r^*$. They also fail to apply for $p = 1$ since if r^* is degenerate, r^* must equal f on the endpoints and r^* of degeneracy ≥ 2 cannot be best to $f \not\equiv r^*$. References are given in Dunham (1974). The situation for $p < 1$ is open.

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