КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 328—333

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 328-333

SUPERCONVERGENCE IN FINITE ELEMENT METHOD FOR A DEGENERATED BOUNDARY VALUE PROBLEM

M.EL HATRI

1. INTRODUCTION. The error estimate of Galerkin-finite element method for partial equation has an optimal order in the energy norm However, this order may be improved in some discrete norms. This phenomenon is called "superconvergence" and it was observed by inginers when strees computationwas applied to certain points.

The error estimate of type superconvergence is analyzed by several authers. For instance, Zlamal[8] has proved a superconvergence of the gradient in the so called Gaussian points when solving a 2^{nd} order elliptic problem by quadrilateral finite elements. Also, by using different techniques, in[1] and in[4] the same type of estimate is established in the midpoints of the sides of the linear triangular elements.

In this paper we analyse the phenomenon of superconvergence of the gradient at the centre of gravity when the linear triangular elements are used for solving a degenerated elliptic problem. To this end we construct a projection operator which renders an account of the singularity of the considered problem. On other hand, because of this singularity, the error estimate is analyzed in the weighted sobolev space introduced in[5] and we obtain, in the discrete L^2-norm of the gradient, an error estimate of order $O(h^2)$ whereas the average convergence rate of the gradient is $O(h)$ (see[7]).

2. PRELIMINARIES AND NOTATIONS. Let Q be a unit square in $\mathbb{R}^2$ and $W_s^{m,2}(Q)$ the real weighted sobolev space introduced in[5], $m \geq 0$ and $m \in \mathbb{N}$:

$$W_s^{m,2}(Q)=\{u\in \mathcal{D}'(Q);\ r_1^s|D^\alpha u|\in L^2(Q), |\alpha|\leq m, \},\ D^\alpha=\frac{\partial^{|\alpha|}}{\partial r_1^{\alpha_1}\partial r_2^{\alpha_2}}, |\alpha|=\alpha_1+\alpha_2, s\in\mathbb{R},$$

α_j -nonnegative integers.

In $W_s^{m,2}(Q)$ we introduce the semi-norm and norm:

$$|u|_{W_s^{m,2}(Q)} = \Big(\sum_{|\alpha|=m} \| r_1^s D^\alpha u\|_{L^2(Q)}\Big)^{1/2}, \quad \|u\|_{W_s^{m,2}(Q)} = \Big(\sum_{k=0}^{m} |u|^2_{W_s^{k,2}(Q)}\Big)^{1/2}$$

Also, we adopt the notation $W_0^{m,2}(Q)=H^m(Q)$, $\|\cdot\|_{W_0^{m,2}(Q)}=\|\cdot\|_{m,Q}$, where $H^m(Q)$ is the usual sobolev space.

We consider the degenerated boundary-value problem:

$$\mathcal{L}(u)= \frac{1}{r_1}\frac{\partial}{\partial r_1}\Big(r_1 d_1(r)\frac{\partial u}{\partial r_1}\Big)+\frac{\partial}{\partial r_2}\Big(d_2(r)\frac{\partial u}{\partial r_2}\Big)= -f(r),\ r=(r_1,r_2)\in Q$$

(1)

$$u|_{\Gamma'}=0,\ \Gamma'=\partial Q\setminus\Gamma_0,\ \Gamma_0=\{(r_1,r_2)\ ;\ r_1=0\}.$$

We assume that $\exists\ C_0=\text{const}>0$ such, that $d_k(r)\geqslant C_0$, $k=1,2$, $\forall\ r\in Q$.

Take into account the condition $\lim\limits_{r_1\to 0}\big(r_1 d_1(r)\frac{\partial u}{\partial r_1}\big)=0$, then multipling $\mathcal{L}(u)$ by $v\in \overset{0}{W}{}^{1,2}_{1/2}(Q)=\{u\in W^{1,2}_{1/2}(Q);\ u|_{\Gamma'}=0\}$ and using the Green's theorem, we obtain the variational problem:

(2) $\quad a(u,v)=(f,v),\ \forall\ v\in \overset{0}{W}{}^{1,2}_{1/2}(Q)$

where $\quad a(u,v)=\sum_{k=1}^{2}\int_Q r_1 d_k\frac{\partial u}{\partial r_k}\frac{\partial v}{\partial r_k}\,dr,\quad (f,v)=\int_Q r_1 f\, v\, dr$.

2. FORMULATION OF THE DISCRETE PROBLEM. Introduce the uniform mesh $\Sigma_h=\{(r_1^i,r_2^i)=a_i\in Q,\ r_k^i=i\,h,\ k=1,2,\ i=0,\dots,N,\ h=\frac{1}{N}\}$, $0<h<1$. And we consider a family of triangulation T_h consisting of the linear triangular elements e with vertices lying on Σ_h and such, that $e\cap\overset{0}{\Sigma}_h\neq\emptyset$, where $\overset{0}{\Sigma}_h=\{a_i\in\Sigma_h,\ a_i\notin\Gamma'\}$. To T_h we associate the finite dimensional subspace $S_h=\{v\in C^0(\bar Q)\ ;\ v|_e\in P_1(e),\ v|_{\Gamma'}=0\}$, where $P_k(e)$ is the class of polynomials of degree $\leqslant k$.

Let $a_i\in\Sigma_h$. There exists a unique function $\varphi_i\in C^0(\bar Q)$, such that:

i) $\forall\ e\in T_h$, $\varphi_i|_e\in P_1(e)$, ii) $\varphi_i(a_j)=\delta_{ij}$.

Observe that S_h is engendered by the basis function φ_i corresponding to the set $\overset{0}{\Sigma}_h$. The Galerkin-finite solution $u_h\in S_h$ is defined by:

(3) $\quad a(u_h,v)=(f,v),\ \forall\ v\in S_h\subset \overset{0}{W}{}^{1,2}_{1/2}(Q)$.

3. SUPERCONVERGENCE THEOREM. In this paragraph, by M we denote a generic positive constante not necessarily the same in any two places which does not depend on h.

Lemma 1. Let $m \in \mathbb{N}, m \geqslant 2$. Then, the following continous injection holds:

$$(4) \qquad W_s^{m,2}(Q) \hookrightarrow C^0(\bar{Q}) \quad , \quad s=0,1/2 .$$

For $s=\frac{1}{2}$, (4) is prouved in[6, th.4.7]. For s=0, it's the sobolev's lemma.

Lemma 2. Let L be a linear bounded functional on $W_s^{k+1,2}(Q), s=0,\frac{1}{2}$, and let it vanishs in $P_k(Q)$. Then, there exists a constant $C=C(Q)$ such that:

$$|L(w)| \leqslant C \, |w|_{W_s^{k+1,2}(Q)} \ , \ s=0,\tfrac{1}{2} \ , \ \forall\, w \in W_s^{k+1,2}(Q) \ .$$

This lemma is a consequence of the following inequality:

$$(5) \qquad \inf_{p \in P(Q)} \|u+p\|_{W_s^{k+1,2}(Q)} \leqslant C|u|_{W_s^{k+1,2}(Q)} \ , \ s=0,1/2 \ .$$

For the proof of (5), for s=0 and $s=\frac{1}{2}$, see[2, p.192]and[6, th.4.6] .

Let $w \in C^1(Q) \cap \overset{0}{W}{}^{1,2}_{1/2}(Q)$. We define the following weighted discrete norm:

$$\|w\|^*_{h,Q} = \Big(\sum_{e \in T_h} h^2 \Big(r_1^{1/2} \frac{\partial w}{\partial r_k}\Big)^2 (b_e^*) \Big)^{1/2} \quad ,$$

where $b_e^*=(r_1^*,r_2^*)$ is a centre of gravity of mass, distributed over e with density r_1 i.e. $r_j^* = \int_e r_1 . r_j \, dr . p_e^{-1}$, $j=1,2$, $p_e = \int_e r_1 \, dr$.

Lemma 3. There exists an operator $\Pi_h \in \mathcal{L}(W^{1,2}_{1/2}(Q);S_h)$ such, that:

$$\| u - \Pi_h u \|^*_{h,Q} \leqslant M \, h^2 \, \| u \|_{W^{3,2}_{1/2}(Q)} \ , \quad \forall\, u \in W^{3,2}_{1/2}(Q) \ .$$

Proof: Let $e \in T_h$ with vertices denoted by b_k , k=1,2,3. We distinguish two cases.

a) $e \cap \Gamma' = \emptyset$. Let $\Pi_e \in \mathcal{L}(W^{1,2}_{1/2}(e);P_1(e))$ be a given operator. If we consider an adjacent element e' to $e, e' \in T_h$, with a common side$[b_\mu, b_\nu]$ then, to e' we correspond an operator $\Pi_{e'} \in \mathcal{L}(W^{1,2}_{1/2}(e);P_1(e))$ defined by:

$$\Pi_{e'} w(r) = \Pi_e w(b_\mu) + \int_{e'} r_1 \|l\| \, D\, w(r)(l) \, dr . p_{e'}^{-1} . \varphi_\nu |_{e'} \ ,$$

$$\Pi_e w(b_\mu) = \Pi_e w(b_\nu) - \int_{e'} r_1 \|l\| \, D\, w(r)(l) \, dr . p_{e'}^{-1} \quad ,$$

where $\varphi_{\nu|e}(b_\nu)=1$, $\|\cdot\|$ -euclidean norm, $l=\overrightarrow{b_\mu b_\nu}$, $D\,w(r)(l)$ -directional derivative along the side $[b_\mu, b_\nu]$.

b) $e\cap\Gamma'\neq\emptyset$. In this case we put:

$$\Pi_e w(r) = \int_e r_1 \|l\| D\,w(r)(l)\,dr.p_e^{-1}\cdot \varphi_{|e},$$

$$\Pi_e w(b) = 0\ ,b \text{ is an arbitrary vertex of } e \text{ lying on } \Gamma',$$

where $\varphi_{|e}$ is a basic function crresponding to a node of $\overset{\circ}{\Sigma}_h\cap e$.

In this way we have constructed, consecutively, on each element $e\in T_h$ an operator $\Pi_e\in\mathcal{L}(W^{1,2}_{1/2}(e);P_1(e))$ such, that:

$$\text{i)}\quad \int_e r_1\sum_{k=1}^{2}\frac{\partial}{\partial r_k}(w-\Pi_e w)\frac{\partial v}{\partial r_k}\,dr = 0,\ \forall v\in P_1(e),\ \forall w\in W^{1,2}_{1/2}(e)$$

$$\text{ii)}\quad \Pi_e w(b_k) = \Pi_{e'}w(b_k)\ ,\ k=\mu,\nu;\ \mu,\nu=1,2,3\ .$$

Now, we let $\Pi_h\in\mathcal{L}(W^{1,2}_{1/2}(Q);S_h)$ be the following operator:

$$\Pi_h w(r) = \sum_{k=1}^{M}\Pi_k w(a_k)\,\varphi_k(r)\ ,$$

where $\Pi_k\equiv\Pi_{e_k}$, $a_k\in\overset{\circ}{\Sigma}_h\cap e_k$, $e_k\in T_h$, $M=\operatorname{card}\overset{\circ}{\Sigma}_h$

It's not difficult to verify the following properties:

1) $\forall e\in T_h$, $\Pi_h w|_e = \Pi_e w$, 2) $\Pi_h w\in S_h$, $\forall w\in W^{1,2}_{1/2}(Q)$.

Further, we pass to estimate the expression:

$$(6)\qquad (\operatorname{mes} e)^{1/2}\Big(r_1^{1/2}\frac{\partial}{\partial r_k}(w-\Pi_h w)\Big)(b_e^*)\ ,\ k=1,2$$

We consider the reference finite element $\hat{e}=\{\hat{r}=(\hat{r}_1,\hat{r}_2)\in\mathbb{R}^2, \hat{r}_1+\hat{r}_2\leqslant 1\}$

There exists a mapping $F: r_k=(\hat{r}_k+i)\,h$, $k=1,2$ such, that $F(\hat{e})=e$, $\forall e\in T_h$

We adopt the notation $F(\hat{r})=r$, $w(r)=w(r_1(\hat{r}),r_2(\hat{r}))=\hat{w}(\hat{r})$.

Consider the functional $L(\hat{w})=h(r_1^*)^{1/2}\Big(\frac{\partial\hat{w}}{\partial\hat{r}_1}-\frac{\partial\widehat{\Pi_h w}}{\partial\hat{r}_1}\Big)(\hat{r}_1^*,\hat{r}_2^*)$, where $\hat{r}_k^*=\int_{\hat{e}}r_1\cdot\hat{r}_k\,d\hat{r}.p_{\hat{e}}^{-1}$, $p_{\hat{e}}=\int_{\hat{e}}r_1\,d\hat{r}$, $k=1,2$. By using F, it follows

$L(\hat{w})=(r_1^*)^{1/2}\Big(\frac{\partial\hat{w}}{\partial\hat{r}_1}-\int_{\hat{e}}r_1\frac{\partial\hat{w}}{\partial\hat{r}_1}\,d\hat{r}.p_{\hat{e}}^{-1}\Big)(\hat{r}_1^*,\hat{r}_2^*)$. Therefore, $L(\hat{w})=0$, $\forall\hat{w}\in P_2(\hat{e})$

We distinguish two cases.

a) $e\cap\Gamma_0\neq\emptyset$ /$i=0$/. Hence, $L(\hat{w})=(h\hat{r}_1^*)\Big(\frac{\partial\hat{w}}{\partial\hat{r}_1}-\int_{\hat{e}}r_1\frac{\partial\hat{w}}{\partial\hat{r}_1}\,d\hat{r}.p_{\hat{e}}^{-1}\Big)(\hat{r}_1^*,\hat{r}_2^*)$

Then, the lemma 1./s=1/2/ yields:

$$|L(\hat{w})| \leqslant M\ h^{1/2} \|\hat{w}\|_{W^{3,2}_{1/2}(\hat{e})}\ ,\forall\,\hat{w} \in W^{3,2}_{1/2}(\hat{e})\ .$$

b) $e \cap \Gamma_0 = \emptyset$ /$i \geqslant 1$/ .In this case we have, by using the lemma 1./s=0/

$$|L(w)| \leqslant M\ h^{1/2}(1+i)^{1/2}\|\hat{w}\|_{3,\hat{e}}\ ,\forall\,\hat{w} \in H^3(\hat{e})\ .$$

Now, applying the lemma 2. and using the mapping F, we get:

$$|L(w)| \leqslant \begin{cases} M\ h^{1/2}|\hat{w}|_{W^{3,2}_{1/2}(\hat{e})} = M\ h^2 |w|_{W^{3,2}_{1/2}(e)}\ ,\ e\cap\Gamma_0 \neq \emptyset\ , \\ M\ h^{1/2}(1+i)^{1/2}|\hat{w}|_{3,\hat{e}} \leqslant M\ h^2(h(1+i)/hi)^{1/2}|w|_{W^{3,2}_{1/2}(e)}\ ,\ e\cap\Gamma_0 = \emptyset \end{cases}$$

In the same way, we estimate the rest term of (6) .

Theorem. Let $u \in W^{3,2}_{1/2}(Q)$ and $d_k \in C^1(\bar{Q})$, k=1,2. Then:

$$(7)\qquad \|u - u_h\|^*_{h,Q} \leqslant M\ h^2\ \|u\|_{W^{3,2}_{1/2}(Q)}\ .$$

Proof: From (2) and (3), it follows $a(u_h - \Pi_h u, v) = a(u - \Pi_h u, v)$ .Hence, because $d_k(r) = d_k(a_i) + O(h), a_i \in e \cap \Sigma_h, r \in e$:

$$(8)\quad a(u_h - \Pi_h u, v) = \sum_e \sum_{k=1}^{2} \int_e r_1 \frac{\partial}{\partial r_k}(u - \Pi_h u)\frac{\partial v}{\partial r_k}\,dr\ +$$
$$\sum_e \sum_{k=1}^{2} \int_e r_1 O(h)\frac{\partial}{\partial r_k}(u - \Pi_h u)\frac{\partial v}{\partial r_k}\,dr$$

The first sum of the right-hand of (8) vanishs, because the propertie i) (see the proof of lemma 3). Proceeding as befor (see the estimate of the functional L), for estimate the rest sum of (8). We get

$$(9)\quad a(u_h - \Pi_h u, v) \leqslant M\ h^2\ \|u\|_{W^{3,2}_{1/2}(Q)}\cdot|v|_{W^{1,2}_{1/2}(Q)}\ ,\ \forall u \in W^{3,2}_{1/2}(Q), \forall v \in S_h.$$

On the other hand, it's easy to see that :

$$(10)\qquad \|\Pi_h u - u_h\|^*_{h,Q} \leqslant M\ |\Pi_h u - u_h|_{W^{1,2}_{1/2}(Q)}\ .$$

Hence, puting in (9) $v = u_h - \Pi_h u$ and using (10) and the condition $d_k(r) \geqslant C_0$, we obtain:

$$\|\Pi_h u - u_h\|^*_{h,Q} \leqslant M\ h^2 \|u\|_{W^{3,2}_{1/2}(Q)}\ ,\forall u \in W^{3,2}_{1/2}(Q).$$

Finally, by appling the triangle inequality, we get (7) .

Remark 1. Successfully,we can construct a similar operator in order to obtain a superconvergence of the gradient at some points lying on the sides of each elements.Therefore,the obtained result in[3] can be ameliorated.

Remark 2. If the family T_h are nonuniform,the constructed operator $\prod_h$ in §3 permits also to obtain the result of the preceeding theorem.

ACKNOWLDGEMENTS. The auther wishes to express his sincere appreciation to professor R.LAZAROV for his very valuable suggestions.

Reference

1. A.Andreev. Compt.Rend.Acad.Bulg.Sci.37,1984,9.
2. P.G.Ciarlet.The finite element method for elliptic problem,North-Holland,1976.
3. M.El Hatri.Compt.Rend.Acad.Bulg.Sci.36,1983,12.
4. M.Křížek,P.Neittaanmäki.University of Jyväskylä,Preprint 24,1983.
5. A.Kufner.Weighted Sobolev Space,Leibnitz,Teubner-Verlagsgsellschaft,1980,1.
6. B.Mercier,G.Raugel.RAIRO Anal.Num.16,1982,4.
7. S.Z.Zhou.SIAM J.Num.Anal.20,1983,5.
8. M.Zlamal,Lecture Notes,1975,606.

Institute of Mathematics with Computer Center
Bulgarian Academy Sciences
P.O.Box 373 1090 Sofia Bulgaria