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TAUBERIAN THEOREMS ON GROUPS AND BANACH MODULES

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1. Introduction. It is the purpose of this note to present several general versions of Tauberian Theorems in the spirit of Wiener's work. As will be seen it is natural to formulate these results in the setting of lca (locally compact Abelian) groups, although of course the special cases arising by considering the m-dimensional Euclidean space or even the real line will be the most interesting ones. As in Wiener's work one of the basic tools will be the Theorem of Wiener-Levy (on the local inversion of Fourier transforms) which is well known to hold true for Beurling algebras $L^1_w(G)$. It will be combined with results on convolutive structures between certain triples of Banach function spaces on groups, in particular with basic results concerning Segal algebras and their weighted analoga (Banach ideals in a Beurling algebra). As one has to work with multipliers, i.e. with operators which commute with convolutions, it is convenient to explain the general frame in the setting of Banach modules. Among the concrete applications results involving the socalled Wiener-type spaces (defined by means of uniform decompositions, for a description see [14], [21], [1], [4] and others), or other decomposition spaces (cf. e.g. [9]) will be the most interesting ones. Using such spaces one obtains among others counterparts and refinements of classical results due to Wiener and Bochner/Chandrasekharan, (cf. [33], [34], [2], or Wiener's collected work, Vol. II), or results as conjectured by Edwards ([6]) or Johnson ([23], p.132), for example. For general informations concerning Wiener's work on Tauberian theorems and its generalizations cf. [33], [34], [31], [30], [6] and of course the collected papers of N. Wiener, including various comments on them ([35]).

2. Notation and preliminaries. Our general setting concerns Banach spaces of measures or distributions on a lca group G. We shall stick as far as possible to the terminology used in [31] or earlier publications

by the author. In particular, $\mathcal{K}(G)$ denotes the space of continuous complex-valued functions on G with compact support. Endowed with its natural inductive limit topology it has $R(G)$, the space of Radon measures as its topological dual. The subspace $M(G)$ of bounded measures will be identified with the Banach dual of $C^0(G)$, the closure of $\mathcal{K}(G)$ in $(C^b(G), \| \|_\infty)$. $C^{ub}(G)$, the space of uniformly continuous functions on G coincides with $L^1(G) * L^\infty(G)$ (cf. [20]). The Lebesgue spaces $(L^p(G), \| \|_p)$ of (classes of) p-integrable functions with respect to Haar measure dx are embedded into $R(G)$ (with the vague topology), or even the closed subspace $L^1_{loc}(G)$, which is endowed with the family of seminorms

$$f \to \int_K |f(x)|\,dx .$$

$(L^1(G), \| \|_1)$ is an ideal in the Banach algebra $M(G)$ with respect to convolution. A Banach space is called a <u>BF-space</u> on G if it is continuously embedded into $L^1_{loc}(G)$. It is called <u>solid</u> if the implication

$$f \in B,\ g \in L^1_{loc},\ \text{and } |g(x)| \leq |f(x)| \text{ a.e.} \Rightarrow g \in B \text{ and } \|g\|_B \leq \|f\|_B$$

holds true. Occasionally also Banach spaces of distributions will arise. These will always be spaces in <u>"standard situation"</u> as treated in [3], for example, and it will be justified to think in terms of tempered distributions in the case of $\mathbb{R}^m$.

The basic operations (inversion, translation, and convolution) are defined on $\mathcal{K}(G)$ as follows (and extended to the more general setting of distributions by continuity or by transposition):

$$f^\vee(x) := f(-x); \quad L_y f(x) := f(x-y); \quad f * g(x) := \int_G f(x-y)g(y)dy.$$

The characters, i.e. the continuous homomorphisms from G into the unit circle are denoted by χ_t. For short we write $\chi_t(x) = \langle t,x\rangle = \langle x,t\rangle$. They form the dual group $\hat{G}$ under multiplication.

Recall that $(\mathbb{R}^m)^\wedge$ may be identified with $\mathbb{R}^m$ as a topological group by the correspondence $t \leftrightarrow \chi_t : x \to \exp(i\, xt)$, $x,t \in \mathbb{R}^m$. We write M_t for the multiplication operator $f \to \chi_t . f$. A BF-space (or a Banach space of distributions) is called translation or character invariant if one has $L_y B \subseteq B$ for $y \in G$ or $M_t B \subseteq B$ for all $t \in \hat{G}$, respectively. The operator norms on B are denoted by $\|L_y\|_B$ and $\|M_t\|_B$ respectively. If they act isometrically on B one speaks of isometric invariance. A BF-space is called a <u>homogeneous Banach space</u> (cf. [24]) if it is isometrically translation invariant and has <u>continuous translation</u> (i.e. $\lim_{y \to e} \|L_y f - f\|_B = 0$ for all $f \in B$). If furthermore, B is a dense subspace of $L^1(G)$ it is called a <u>Segal algebra</u> (cf. [31], Chap. 6, § 1).

By means of vector-valued integration one shows that any homogeneous Banach space is a Banach module with respect to convolution over $L^1(G)$. In particular, any Segal algebra is an ideal in $L^1(G)$. The same device implies that, more generally, $(B,\|\ \|_B)$ is a Banach convolution module over some Beurling algebra $L^1_w(G)$, defined as $L^1_w(G) := \{f \mid f\, w \in L^1(G)\}$ (here w may be assumed to be a continuous, strictly positive weight function on G satisfying $w(x+y) \leq w(x)w(y)$ for all $x,y \in G$, cf. [10]), if translation is continuous in B, or if B is the dual of a space having continuous translation (such as $L^\infty(G)$). A typical choice for w is

$$w(y) := \max(1, \|L_y\|_B), \text{ (cf. [7], § 2).}$$

An operator $T: B \to B^1$ between two such spaces is called a multiplier (module homomorphismus) if $T(k * f) = k * Tf$ for $k \in \mathcal{K}(G)$, $f \in B$. If translation is continuous in B this is the case if and only if $TL_y = L_y T$ for all $y \in G$ (cf. [26], [7]).

For bounded measures the Fourier transform is given by $\hat{\mu}(t) := \mu(\chi_t)$. For distributions it may be characterized by the formula $\langle\hat{\sigma},k\rangle = \langle\sigma,\hat{k}\rangle$ for suitable test functions k on $\hat{G}$ ($G = \hat{G}^\wedge$!). We mention that for any lca group the Fourier transform is well defined (as a quasimeasure, with test functions in $\mathcal{K}(\hat{G}) * \mathcal{K}(\hat{G})$, cf. [12]) on any homogenous Banach space. Given such a space B, $\hat{B}$ denotes the space of Fourier transforms of its elements which is always endowed with its canonical norm $\|\hat{f}\|_{\hat{B}} := \|f\|_B$. Observe that B is translation (character) invariant if and only if $\hat{B}$ is character (translation) invariant, and B is a $L^1_w G$ convolution module if and only if $\hat{B}$ is a pointwise module over $A_w(\hat{G}) := (L^1_w)^\wedge$. Since $A_w(\hat{G})$ is a Wiener algebra in the sense of Reiter (cf. [31], Chap. II : i.e. a regular topological algebra of continuous functions, with the compactly supported functions in it as a dense ideal and satisfying the principle of local inversion, that is: for any compact set $M \subseteq \hat{G}$ and $h \in A_w$ with $h(t) \neq 0$ for $t \in K$ there exists $g \in A_w$ such that $g(t) = 1/h(t)$ for all $t \in K$) if and only if w satisfies the condition of Beurling-Domar

$$\text{(BD)}: \sum_{n=1}^{\infty} n^{-2} \log w(nx) < \infty \text{ for all } x \in G),$$

we shall consider only such weights (cf. [5] or [31], Chap. 6, § 3.1). It is obvious that polynomial weights on $\mathbb{R}^m$, such as $w_\alpha(x) = (1 + |x|)^\alpha$ (cf. [31], Chap. 1, § 6.1) satisfy this condition for $\alpha \geq 0$. For them and other related weights on $\mathbb{R}^m$ the principle of local inversion in $A_w(\mathbb{R}^m)$ can be proved directly (without recurrence to Gelfand theory) by an elegant direct device for the proof of the Wiener-Levy theorem (cf.

[31], Chap. 1, § 6.5).

Finally we shortly recall a possible description of Wiener type spaces (cf. [14]). We let $(B,\| \|_B)$ be a Banach space of distributions (cf. [3], § 2) which is a pointwise Banach module over some (regular) algebra $A_w(G)$ (e.g. B is a solid BF-space), and $(C,\| \|_C)$ be any solid L^1_w-Banach convolution module (such as $L^q_w(G)=\{f \mid f\,w \in L^q\}$, with the norm $\|f\|_{q,w} := \|fw\|_q$, for example). Then the Wiener type space $W(B,C)$, with local component B and global component C may be described as (fixing any nonzero test function $\varphi \in A_w \cap \mathcal{K}(G)$):

$$W(B,C) := \{f \mid f \in B_{loc}, \text{ such that } f^{(B)}: x \to \|(L_x\varphi)f\|_B \in C\}.$$

It is a Banach space with the norm

$$\|f\|_{W(B,C)} := \|f^{(B)}\|_C.$$

An alternative description involves bounded, uniform partitions of unity (cf. [11], [14]), e.g. for $G=\mathbb{R}^m$ one has (fixing any sufficiently large ball or cube Q).

$$W(L^p,L^q_w) := \{f \mid f \in L^p_{loc}, \|f\|_w := [\Sigma_{n\in\mathbb{Z}}\, w^q(n).(\int_{n+Q} |f(y)|^p dy)^{p/q}]^{1/q} < \infty\}.$$

For papers using such descriptions cf. [21], [1], [4], [34], [19] and others. The spaces $W(L^p,L^1)$ are Segal algebras (cf. [11], [25], [8], [19], [13], for example).

As a basic fact one has the elementary formula for duality (if $\mathcal{K}(G)$ is dense in $W(B,C)$):

$$W(B,C)' = W(B',C')$$

(cf. [16] for the result in its full generality), and the convolution relation

$$W(B^1,C^1) * W(B^2,C^2) \subseteq W(B^3,C^3),$$

if $B^1 * B^2 \subseteq B^3$ and $C^1 * C^2 \subseteq C^3$ (cf. [14], Theorem 3 , and [1], [3], [21] for special cases).

Finally we mention that assertions for weighted L^p-spaces (as C^i above) in this direction are to be found in [10], for example. Other decomposition spaces (dyadic ones) and corresponding results are treated in [9] (see also [16] for the general philosophy).

As the L^1-version of Wiener's Tauberian Theorem (Th. 4 in [34]) is available for any lca group G, (cf. [31], Chap. 6) let us start with the following result (including Wiener's Theorem 5 as well as several of its generalizations, e.g. [22]) which may be considered as an almost immediate application of basic results concerning Segal algebras.

Theorem 1. Let B be any homogeneous Banach space on G such that $B \cap L^1$ is a dense subspace of B as well as of L^1. Let $\sigma \in B'$ be given and $f_o \in B \cap L^1$, such that $\sigma * \check{f}_o \in C^o(G)$. If $\hat{f}_o(t) \neq 0$ for all $t \in G$ then $\check{f} * \sigma \in C^o(G)$ for any $f \in B$.

Proof. Since $T_\sigma : f \to \check{f} * \sigma$ $(T_\sigma f(x) := \langle \sigma, L_x f \rangle)$ defines a multiplier (continuous linear operator commuting with translations) from B to $C^b(G)$ it is clear that

$$I_\sigma := \{ f \mid f \in B \cap L^1,\ T_\sigma f \in C^o(G) \}$$

is a closed, translation invariant subspace (hence ideal) of the Segal algebra $(B \cap L^1, \| \|_B + \| \|_1)$, and $f_o \in I_\sigma$. By Wiener's approximation theorem I_σ is dense in $L^1(G)$, but the ideal theorem for Segal algebras (cf. [31], Chap. 6, § 2.5) implies that $I_\sigma = \bar{I}_\sigma^{L^1} \cap (L^1 \cap B) = L^1 \cap B$. The density of $L^1 \cap B$ in B together with the continuity of T_σ then implies the result in full generality.

Remark 1. Observe that this result applies to any Segal algebra or to any solid homogeneous Banach space B containing $\mathcal{K}(G)$ as a dense subspace.

Remark 2. It is clear from the proof that it would be possible to replace $C^o(\mathbb{R}^m)$ by any other closed, translation invariant subspace of $L^\infty(\mathbb{R}^m)$, e.g. $AP(\mathbb{R}^m)$ [or $WAP(\mathbb{R}^m)$], the space of [weakly] almost periodic functions on $\mathbb{R}^m$. For B being the Segal algebra $W(L^q, L^1)$ one might than choose S^p-a.p., the Stepanoff almost periodic functions, as a closed invariant subspace of $W(L^p, L^\infty) = W(L^q, L^1)'$ (cf. [15]).

Although it would now be possible to give a long list of special cases and/or variants of our first theorem we prefer to direct our attention directly to an abstract variant which should extract the general structure from as many "general Tauberian theorems" as possible. This can be done (cf. Theorems 2 and 4 below) using the following terminology:

Definition. A quintuple (A,B^1,B^2,B^3,B^4) (for short $(A,\mathfrak{B})$) of Banach spaces will be called a Tauberian system on G if the following conditions hold true:

I) $(A,\|\ \|_A)$ is a Banach algebra with respect to convolution;

II) The spaces B^i, $1 \le i \le 4$ are Banach convolution modules over A, and B^4 is a closed submodule of B^3;

III) (B^1,B^2,B^3) is a Banach convolution triple, i.e. $B^1 * B^2 \subseteq B^3$, or more precisely: There is a continuous embedding of B^2 into the space of multipliers $H_G(B^1,B^3)$.

Theorem 2. Let $(A,\mathfrak{B})$ be a Tauberian system. Assume that $f_o \in B^1$ is a cyclic vector for B^1 (i.e. that $A * f_o$ is dense in B^1), and let $g \in B^2$ be given such that $f_o * g \in B^4$. Then one has $f * g \in B^4$ for all $f \in B^1$.

Proof. By assumption, given $f \in B^1$ there exists a sequence $(a_n)_{n\geq 1}$ in A such that $f = \lim_{n\to\infty} a_n * f_o$. It follows that

$$f * g = \lim_{n\to\infty} (a_n * f_o) * g = \lim_{n\to\infty} a_n * (f_o * g) \text{ in } B^3.$$

Since $f_o * g \ \ B^4$ by the assumption and because B^4 is a closed submodule of B^3 it follows therefrom easily that $f * g \in B^4$ for all $f \in B^1$.

Applications of this result will be given at the end of this paper. In order to point out that there are situations, quite similar to Theorem 1, where our abstract result is not applicable, but which still seems to have a "Tauberian character" we state the following (essentially wellknown) Lemma:

Lemma 3. For $\mu_o \in M(G)$ the following conditions are equivalent:

i) $\hat{\mu}_o(t) \neq 0$ for all $t \in \hat{G}$;(" μ_o has nonvanishing Fourier transform")

ii) $\{L_y \mu_o \mid y \in G\}$ is a total subset in M(G) for the weak topology $\sigma(M(G), C^{ub}(G))$;

iii) Given $f \in C^{ub}(G)$ and any closed, bounded subspace D of $L^\infty(G)$, (e.g. $C^o(G)$ or $\{0\}$) $\mu_o * f \in B$ implies $\mu * f \in B$ for all $\mu \in M(G)$.

It motivates our next result. Recall that the essential part B_e of an A-Banach module is defined as the closed linear span of AB in B. B is called an essential A-module if $B = B_e$. For our applications it will be important to observe that a translation invariant Banach module over a solid, translation invariant Banach convolution algebra A, containing $\mathcal{K}(G)$ as a dense subspace, is an essential Banach module over A if and only if translation is continuous in B (This can be shown as usual by vector-valued integration, using for the converse the fact that translation is continuous in A, cf. [7], [32] for related generalities).

Theorem 4. Let $(A,\mathfrak{B})$ be a Tauberian system on G. Assume that translation is continuous in B_2 and that A contains $\mathcal{K}(G)$ as a subspace (i.e. essentially $B^2 = B^2_e$ is assumed). If then $f_o * g \in B^4$ for some $g \in B^2$ it follows $f * g \in B^4$ for all $f \in B^1$ if f_o is a cyclic vector for B^1_e only, i.e. if $f_o \in H_G(A,B^1)$ has the property that $A * f_o$ is a dense subspace of B^1_e (e.g. $f_o \in B^1$, such that the space $A * f_o$ is dense in B^1_e).

Proof. Continuity of translation in B^2 implies (that B^2 is an essential convolution module over some Beurling algebra and) that for $g \in B^2$, $f \in B^1$ there exists $k \in \mathcal{K}(G)$ $(\subseteq A \cap L^1_w(G))$ such that $\|k * g - g\|_{B_2} < \varepsilon / \|f\|_{B^1}$. According to our assumption $f * k = k * f \in B^1_e$ can be approximated as follows:

$$\|k * g - a * f_o\|_{B^1} < \varepsilon / \|g\|_{B^2} \text{ for some } a_o \in A.$$

It then follows

$$\begin{aligned} \|f * g - a * f_o * g\|_{B^3} &\leq \|f * g - f * k * g\|_{B^3} + \|(k * f - a * f_o) * g\|_{B^3} \\ &\leq C[\, \|f\|_{B_1} \; \|k * g - g\|_{B_2} + \|k * f - a * f_o\|_{B^1} \|g\|_{B^2}] \\ &\leq 2C\varepsilon . \end{aligned}$$

Since $a * (f_o * g)$ belongs to the closed submodule B^4 of B^3 the Theorem is proved. (If A has bounded approximate units, e.g. if A is a Beurling algebra, then a more elegant argument can be given, using the factorization theorem, cf. [20]).

Remark 3. The essential conclusion ((i) ⇒ (iii)) in Lemma 3 actually arises, again as a consequence of the Wiener-Levy theorem, if one considers the Tauberian system $(L^1, M, C^{ub}, C^{ub}, D)$. (Recall that a closed translation invariant subspace D of C^{ub} is a homogeneous Banach space, hence an L^1-module for convolution!). It may therefore also considered as a special case of the following slightly more concrete result:

Theorem 5. Let A be a solid, translation invariant Banach convolution algebra in $L^1(G)$, containing $\mathcal{K}(G)$ as a dense subspace (subalgebra). If $\hat{A}$ admits local inversion and if A is the Banach dual of another Banach space of measures on G, then one has (writing A'_o for the closure of $\mathcal{K}(G)$ in A' and $A'_c := \{\mu \mid \mu \in A', \lim_{y\to 0} \| L_y\mu - \mu\|_{A'} = 0\}$):

If a multiplier $\mu_o \in H_G(A'_o, A'_o)$ satisfies $\hat{\mu}_o(t) \neq 0$ for all $t \in \hat{G}$, then $\mu_o * h \in A'_o$ for some $h \in A'_c$ implies $\mu * h \in A'_o$ <u>for all</u> $h \in A'_c$.

Remark 4. Any multiplier $\mu_o \in H_G(A,A)$ which can be approximated in the strong operator topology by convolution operators with compactly supported kernels defines an element of $H_G(A'_o, A'_o)$. In fact, the action of μ_o on A', obtained by transposition, leaves A'_o invariant and μ_o may therefore be considered as a multiplier on A'_o.

Proof of Theorem 5. We intend to apply essentially Theorem 4 by first checking that $(A, H_G(A'_o, A'_o), A'_c, A'_c, A'_o)$ is a Tauberian system. Only condition III) deserves discussion in this case.

Let μ be given. Since A is a dual standard space (cf. [3], as a double module over $C^o(G)$ and some Beurling algebra $L^1_w(G)$) we have $\tilde{A} = (A'_o)'_o = (A'_o)'$ (cf. [3], Theorem 3.8, Corollary 3.10). By successive transposition one derives that for μ under consideration

$\mu \in H_G(A'_o, A'_o) \subseteq H_G(A,A) \subseteq H_G(A',A') \subseteq H_G(A'_c, A'_c)$, because $A'_c = L^1_w * A'$. Thus III) is satisfied, and of course one has density of

$$A * A'_c \supseteq \mathcal{K}(G) * A'_c \text{ in } A'_c.$$

Instead of checking formally that μ_o is cyclic in the required sense if it has nonvanishing Fourier transform we prefer to show directly that, given $k \in \mathcal{K}(G)$ and $h \in H_G(A'_c, A'_c)$ one can approximate (in the operator norm on A'_c) $k * h$ by elements of the form $a * \mu_o$. Interpreting k as element of $L^1_w(G)$ (acting on A'_c of course) it is possible to replace k by $k' \in L^1_w$ such that supp $\hat{k}'$ is compact. But then there exists $g \in A$ such that $\hat{g}(t) = 1/\hat{\mu}_o(t)$ on supp $\hat{k}'$ and thus

$$k' * h = \mu_o * g * k' * h \subseteq A * \mu_o * h \subseteq A * A'_o \subseteq A'_o.$$

The following special case of Theorem 4 is widely applicable:

<u>Theorem 6.</u> Let $(B,\| \|_B)$ be a translation invariant BF-space on $\mathbb{R}^m$, with continuous translation and such that there exists $\alpha \geq 0, C > 0$ with $\|L_y\|_B \leq C(1+|y|)^\alpha$ for all $y \in \mathbb{R}^m$. Write $M_\alpha(\mathbb{R}^m)$ for the Banach algebra (for convolution) of bounded measures, given by the condition

$$\|\mu\|_\alpha := \int_{\mathbb{R}^m} (1+|y|)^\alpha \, d|\mu|(y) < \infty ,$$

and let D be a closed, translation invariant subspace of B. Then one has: If $g \in B$ satisfies $\mu_o * g \in D$ for some $\mu_o \in M_\alpha(\mathbb{R}^m)$ having nonvanishing Fourier transform, then one has $\mu * g \in D$ <u>for all</u> $\mu \in M_\alpha(\mathbb{R}^m)$.

<u>Proof</u>. One observes that, for the weight function $w_\alpha(y) := (1+|y|)^\alpha$, the quintuple $(L^1_{w_\alpha}, M_\alpha, B, B, D)$ is a Tauberian system satisfying the conditions for $\mu_o = f_o$. Actually, B (and D as a closed subspace) is not only a module over $L^1_{w_\alpha}$, but also over the bigger algebra M_α (which coincides with the algebra $H_{\mathbb{R}^m}(L^1_{w_\alpha}, L^1_{w_\alpha})$; via vector-valued integration). As a consequence of the Theorem of Wiener-Levy for Beurling algebras it follows that $(L^1_{w_\alpha})^\wedge$ is a Wiener algebra in the sense of Reiter (cf. [31], Chap.II and Chap.I, §6.5) the measure μ_o with nonvanishing Fourier transform is cyclic in $(M_\alpha)_e = L^1_{w_\alpha}$.

As a typical choice for B above one might choose $B = W(L^q, L^\infty_w)$, for any $q, 1 \leq q < \infty$, and $D := W(L^q, C^o_w)$, for $w := w_b$, for some $b \in [-a,a]$. As in the L^1-case it is possible to replace D (the subspace of elements vanishing at infinity) by $\mathbb{C}+D$ (the direct sum with the constant functions), in order to obtain assertions concerning generalized limits (cf. [31], Chap.I, §4.5).

We conclude this note with a typical application (out of many other possible applications) of Theorem 2. As will be seen, it may be considered as a generalization of Theorem 83 (Chap VI, §3 in [2]) of Bochner/Chandrasekharan. In order to facilitate comparison we use the following notations for the formulation of our result: For $\gamma > 0$ K_γ denotes the space of continuous functions which are $O(|x|^{-\gamma})$ for $|x| \to \infty$ on $\mathbb{R}^m$ (i.e. $= C^b_{w_\gamma}(\mathbb{R}^m)$) and assume that ψ is a continuous, moderate function (this corresponds to conditions (2.2) and (2.2)" of [2], p.175): For any compact set $K \subseteq \mathbb{R}^m$ there exists exists $C_1, C_2 > 0$ such that $C_1 \leq \psi(x)/\psi(x+y) \leq C_2$ for all $x \in \mathbb{R}^m$, $y \in K$. It is called to be of class ψ_γ' (the prime indicating that condition (2.2)' is not needed), if $w^{-1}_{\gamma+1} * \psi \leq C\psi$ (or equivalently: $K_{\gamma+1} * L^\infty_v \subseteq L^\infty_v$, for $v := \psi^{-1}$). Writing finally $K^o_{\gamma+1}$ for the closure of $\mathcal{K}(\mathbb{R}^m)$ in $K_{\gamma+1} (= C^o_{w_{\gamma+1}})$ one has the following corollary to Theorem 2:

Theorem 7. Let $\delta>m+1$, $\psi \in \Psi_\gamma'$, and a bounded measure $\mu \in W(M,L_v^\infty)(\mathbb{R}^m)$ for $v:=\psi^{-1}$; if for some $g_o \in K_{\gamma+1+\delta}$, having nonvanishing Fourier transfor satisfies

$$\lim_{|x|\to\infty} \psi^{-1}(x)(g_o * \mu)(x) = A \int g_o(x)dx ,$$

then

$$\lim_{|x|\to\infty} \psi^{-1}(x)(f * \mu)(x) = A \int f(x)dx \quad \text{for all } f \in K^o_{\gamma+1} .$$

Proof. As usual one has to consider the case $A=0$ only. One may check that the assumption actually implies that $g_o \in (L^1 \cap C^o)_{w_{\gamma+1}} =: D_\gamma$ (a dense set in $K^o_{\gamma+1}$). Since $w_{\gamma+1}$ is a (submultiplicative) weight and $L^1 \cap C^o(\mathbb{R}^m)$ is an ideal (even Segal algebra) in $L^1(\mathbb{R}^m)$, it follows that D_γ is an ideal in the Beurling algebra $L^1_{w_{\gamma+1}}(\mathbb{R}^m)$ (cf. [10], Satz 3.7), hence D_γ has local inversion. Having continuous translation (because $\mathcal{K}(\mathbb{R}^m)$ is dense) in D_γ the elements h with compact spectrum (support of $\hat{h}$) are dense in D_γ, and it follows as in the proof of Theorem 4 (or cf. Theorem 1) that g_o is a cyclic for D_γ, hence $K^o_{\gamma+1}$. Since the assumptions imply that $(L^1_{w_{\gamma+1}}, K^o_{\gamma+1}, W(M,L_v^\infty), L_v^\infty, C_v^o)$ (the verification of III) is based on Theorem 3 as presented in [14]) is a Tauberian system it is clear that the required conclusion follows from Theorem 2.

Remark 5. Theorem 83 in [2] corresponds to the special case m=1 and $\delta=1$, using the additional condition (2.2)' (p.175). Examples of natural function ψ satisfying the above conditions (for $\mathbb{R}^m$), including anisotropic functions, are easily derived from the material in [10]. It is also clear from the proof that it extends to the setting of (noncompact) lca groups.

Remark 6. In conclusion we mention that a number of Tauberian systems can be obtained from convolution triples as described in [9], [10] or [14] for example. Among the most interesting special cases are perhaps the spaces over dyadic decompositions treated in [9]. One can show that results derived in this way imply Wiener's third Tauberian theorem ([34] Theorem 29, p.177), again as a consequence of Theorem 2 above. A more detailed discussion, including extensions of Wiener's theorem 29 (as studied in [27]-[29], for example) is to be given in [18]. That note also contains generalizations involving the space $M^p(\mathbb{R}^m)$ of functions with bounded means of order p, $1 < p < \infty$. Essential for the possibility of giving an elementary approach to this result will be a characterization of $M^p(\mathbb{R}^m)$ as the dual of a solid Banach convolution algebra that can be characterized by a certain minimality property (cf. [17] for the general approach to such spaces and their basic properties).

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