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ON SOME EXTREMAL PROBLEMS FOR CONJUGATE CLASSES OF FUNCTIONS

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Three types of extremal problems are mainly considered in the approximation theory of functions. They can be formulated in an arbitrary linear normed space X (see [1, p. 11 - 13]) and cover the following cases: 1) approximation of a fixed element $x \in X$ by a fixed set $F \subset X$; 2) approximation of a fixed set $G \subset X$ by means of a fixed set $F \subset X$ and 3) approximation of a fixed set $G \subset X$ by a preassigned class of sets $\{F\} \subset X$. In the books of Lorentz [2], Korneichuk [1], Tihomirov [3] and others, a series of fundamental results is given. They are connected with the precise solution of problems of type 2) and 3) with respect to the best approximation of fixed sets in given spaces.

In the present note we consider only the case of 2π-periodic functions in the classical spaces C and L with their corresponding norms. As techniques of approximation we are going to utilize a subset of trigonometric polynomials of fixed ($n - 1$) degree.

The symbol W^rH^ω ($r = 0,1,\ldots$) denotes a class of 2π-periodic functions, defined in the usual way (see e.g. [1, p. 183]). Everywhere we shall consider $\omega(t)$ as upper convex modulus of continuity complying with the condition:

$$(1) \qquad \int_0^\pi t^{-1}\omega(t)dt < +\infty$$

By $\tilde{f}(x)$ we denote the function, which is trigonometrically conjugate to $f(x)$. So if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

then

$$(2)\quad \tilde{f}(x) = \sum_{n=1}^{\infty} a_n \sin nx - b_n \cos nx = \frac{1}{2\pi}\int_0^{2\pi} f(t)\operatorname{cotg}\frac{x-t}{2}\,dt$$

The existence of the singular integral (2) under sufficiently general conditions for $f(x)$ and the correctness of equality (2) (see [4]) should be noted. Then

$$\widetilde{W^rH^\omega} \overset{def}{=} \left\{ \tilde{f} : f \in W^rH^\omega \right\} \quad (r = 0,1,\ldots)$$

We shall make use of $E_n(.)_X$ as traditional denotion for the best approximation in the space X by means of trigonometric polynomials of $(n-1)$-th degree.

__Theorem.__ Let $\omega(t)$ be an arbitrary upper convex modulus of continuity subjected to condition (1). Then the following precise estimations for the best approximation of the class $\widetilde{W^rH^\omega}$ hold:

$$(3)\quad E_n(\widetilde{W^rH^\omega})_C = \left\| \tilde{f}_{nr} \right\|_C = \frac{1}{2n^r}\int_0^{\pi} \hat{\Phi}_{\pi,r-1}(\pi-t)\,\omega\!\left(\frac{t}{n}\right)dt \quad (n,r = 1,2,\ldots)$$

$$(4)\quad E_n(\widetilde{W^rH^\omega})_L = \left\| \tilde{f}_{nr} \right\|_L = \frac{2}{n^r}\int_0^{\pi} \hat{\Phi}_{\pi,r}(\pi-t)\,\omega\!\left(\frac{t}{n}\right)dt \quad (n,r = 1,2,\ldots)$$

where

$$(5)\quad \hat{\Phi}_{a,r}(x) = \begin{cases} \dfrac{2a^r}{\pi^{r+1}} \displaystyle\sum_{j=0}^{\infty} \frac{(-1)^{jr}\cos(2j+1)\frac{\pi x}{2a}}{(2j+1)^{r+1}}; & 0 \leqq x \leqq a \\[2ex] 0 \quad ; \quad x \geqq a & (r = 0,1,2,3,\ldots) \end{cases}$$

and $f_{nr}(x)$ are the following $2\pi/n$ -periodic (odd and even) recurrently defined functions:

$$f_{nr}(x) = \begin{cases} \displaystyle\int_{\pi/2n}^{x} f_{n,r-1}(t)dt; & r = 1,3,\ldots \\[2ex] \displaystyle\int_{0}^{x} f_{n,r-1}(t)dt; & r = 2,4,\ldots \end{cases} \qquad f_{no}(x) = \begin{cases} \frac{1}{2}\omega(2x) \; ; & 0 \leqq x \leqq \dfrac{\pi}{2n} \\[2ex] \frac{1}{2}\omega\left[2\left(\frac{\pi}{n} - x\right)\right]; & \dfrac{\pi}{2n} \leqq x \leqq \dfrac{\pi}{n} \end{cases}$$

Remarks: 1. In particular, at $\omega(t) = Kt$ ($K = \text{const.} > 0$) we can obtain the result of Favard [5 – 7], Ahiezer and Krein [8] involving uniform metric

$$E_n(\widetilde{W^rKH^1})_C = E_n(\widetilde{W^{r+1}K})_C = K\frac{\widetilde{K}_{r+1}}{n^{r+1}} \quad (n,r=1,2,3,\dots) \tag{6}$$

where

$$\widetilde{K}_r = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^{jr}}{(2j+1)^{r+1}} \qquad (r=1,2,\dots)$$

are the Favard constants.

2. We shall note the important particular case of formula (4) at $\omega(t) = Kt$ ($K = \text{const} > 0$)

$$E_n(\widetilde{W^rKH^1})_L = E_n(\widetilde{W^{r+1}K})_L = K\frac{4}{n^{r+1}}\widetilde{K}_{r+2} \quad (n,r=1,2,\dots) \tag{7}$$

3. For the 2n-1-dimensional diameter of Kolmogorov (see [1, p. 252]) the following upper bounds hold

$$d_{2n-1}(\widetilde{W^rH^\omega};\ C) \leqq E_n(\widetilde{W^rH^\omega})_C = \left\| \widetilde{f}_{nr}\right\|_C \quad (n,r=1,2,\dots)$$

$$d_{2n-1}(\widetilde{W^rH^\omega};\ L) \leqq E_n(\widetilde{W^rH^\omega})_L = \left\| \widetilde{f}_{nr}\right\|_L \quad (n,r=1,2,\dots)$$

The proof of the theorem is long and rather involved. It can be divided into two steps:

1. The lower bounds for $E_n(\widetilde{W^rH^\omega})_X$ ($X = C,L$) can be obtained by the classical method (indicated in [1] at the proof of the similar assertions) after considering the properties of $\widetilde{f}_{nr}$, previously proved.

2. The upper bounds for $E_n(\widetilde{W^rH^\omega})_X$ can be obtained by means of Korneichuk's theory for $\sum$-permutations, which must be modified for unbounded functions. In addition auxiliary classes of functions on the ground of some properties of the extremal functions g_{nr} (see [1, p. 107]) and their conjugate $\widetilde{g}_{nr}$ are to be introduced. Here we shall give only a sketch of our arguments.

a) The case $X = C$. Let $\widehat{D}$ be the set of all summable 2π-periodic functions $g(x)$, which at every point x have finite or in-finite one-sided limits $g(x-0)$ and $g(x+0)$. Denote by $\widehat{D}^r$ the set of r-th 2π-periodic integrals of functions $g \in \widehat{D}$ (g has zero mean value in the period).

Definition. Let $f \in \widehat{D}$. We say that f satisfies the condition σ with a constant a ($a > 0$) and write $f \in \sigma(a)$, if its corrected function $f_1(x)$ (see [1, p. 137]) satisfies the inequality

$$(8) \qquad \sum_{k=1}^{m} \left| f_1(t_k) - f_1(\tau_k) \right| \leqq 4\sigma(x) \quad (0 < x < a)$$

where $\sigma(x) = \sigma(a;x)$ is a preassigned function, annihilating at the origin and strictly increasing in the interval $(0, a)$. The sum in (8) is taken over all non-crossing intervals (t_k, τ_k) from $(x_0, x_0 + 2\pi)$, at the ends of which the function

$$F(x) = \int_{x_0}^{x} f_1(t)dt$$

(with zero average value in the period) satisfies the conditions

$$(9) \quad \tau_k - t_k = x \ ; \ F(t_k) = F(\tau_k) \ ; \ |F(t)| > |F(t_k)| \ (t_k < t < \tau_k)$$

$$k = 1,2,3,\dots m$$

In the capacity of $\sigma(a;x)$ we take

$$(10) \quad \sigma_1 = \sigma_1(a;x) = \frac{1}{2\pi} \ln\left[\operatorname{cotg}\left(\frac{\pi}{4} - \frac{\pi x}{4a}\right)\right] \quad (0 \leqq x < a)$$

$$(11) \quad \sigma_2 = \sigma_2(a;x) = \frac{a}{\pi^2} \sum_{j=0}^{\infty} \frac{\sin(2j+1)\frac{\pi x}{2a}}{(2j+1)^2} \quad (0 \leqq x \leqq a)$$

By $\widehat{D}_{r,r-1}(\sigma_1;\sigma_2;a)$ $(r \geqq 0)$ we denote the set of functions $f \in \widehat{D}^r$, such that $f^{(r)} \in \sigma_1(a)$ and $f^{(r-1)} \in \sigma_2(a)$. In the case $r = 0$ we have $\widehat{D}_{0,-1}(\sigma_1;\sigma_2;a)$ in sense that $f \in \sigma_1(a)$ and

$$F(x) = \int_{c_0}^{x} f(t)dt \in \sigma_2(a)$$

with zero mean value in the period.

Now we can introduce the following classes

$$\widehat{W^rH^n_V} \overset{def}{=} \left\{ g : g \in W^rH^n_V ; \ \widetilde{g} \in \widehat{D}_{r,r-1}(\sigma_1;\sigma_2;\tfrac{\pi}{n}) \right\} \quad (r = 0,1,..;n = 1,2,..)$$

where $W^rH^n_V$ are defined in [1, § 5.1].

For the functional $F_\omega(g)$ [1, § 7.5] it is easy to prove the relation

$$\sup_{g \in W^rH^n_V} F_\omega(g) = \sup_{g \in \widehat{W^rH^n_V}} F_\omega(g) \quad (r = 0,1,..;n = 1,2,...)$$

and we come naturally to the following definition

$$E_n(\widetilde{W^rH^\omega})_C \overset{def}{=} \sup_{f \in H^\omega} \ \sup_{g \in \widehat{W^{r-1}H^n_V}} \int_0^{2\pi} \widetilde{f}(t)g(t)dt \quad (n,r = 1,2,3,...)$$

b) The case $X = L$. At first Kolmogorov's theorem of comparison [1, § 5.6] must be modified for conjugate functions by introducing the classes

$$\widehat{W^r_M(\lambda)} \overset{def}{=} \left\{ f : f \in W^r_M \cap V^r \text{ so that, from } \widetilde{f}^{(r-1)}(a) = \widetilde{\varphi}_{\lambda 1}(\alpha) \right.$$

$$\left. \text{to succeed } \left|\widetilde{f}^{(r)}(a)\right| \leqq \left|\widetilde{\varphi}'_{\lambda 1}(\alpha)\right| \right\} \quad (\lambda > 0 ; r = 1,2,...)$$

at every points a and α, where corresponding derivatives exist. $\varphi_{\lambda r}$ $(r = 0,1,2,...)$ are defined in [1, § 5.4]. By $\widehat{W^rH^n_M}$ $(n,r = 1,2,..)$ we denote the subset of $\widehat{W^r_M(n)}$ whose functions are orthogonal to the trigonometric polynomials of degree $\leqq n-1$. Meanwhile we shall emphasize the result

$$\sup_{g \in \widehat{W^rH^n_M}} \left\| \widetilde{g} \right\|_{L_p} = \left\| \widetilde{\varphi}_{nr} \right\|_{L_p} \quad (1 \leqq p \leqq +\infty ; n,r = 1,2,...)$$

It is easy to prove the correlation

$$\sup_{g \in \widehat{W^rH^n_M}} F_\omega(g) = \sup_{g \in W^rH^n_M} F_\omega(g) \quad (n,r = 1,2,...)$$

and we come to the following definition

$$E_n(\widetilde{W^r H^\omega})_L \overset{def}{=} \sup_{f \in H^\omega} \sup_{g \in \widehat{W^r H^n_M}} \int_0^{2\pi} \tilde{f}(t) g(t)\,dt \quad (n, r = 1, 2, 3, \dots)$$

On the other hand, we can prove the precise equality

$$\sup_{f \in H^\omega} \sup_{g \in \widehat{W^r H^n_M}} \int_0^{2\pi} \tilde{f}(t) g(t)\,dt = \sup_{g \in \widehat{W^r H^n_M}} \sup_{f \in H^\omega} \int_0^{2\pi} f(t)\tilde{g}(t)\,dt.$$

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