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BERSTEIN-TYPE INEQUALITIES WITH RESPECT TO THE VILENKIN--SYSTEMS

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1. Introduction. Let $m=(m_k, k\in\mathbb{N})$ $(\mathbb{N}:=0,1,\ldots)$ be a sequence of natural numbers, the terms of which are not less than 2. Denote Z_{m_k} the discrete cyclic group of order m_k, and define the Vilenkin--group G_m as the direct product of Z_{m_k}'s. Introduce the product topology in G_m. The normalized Haar measure on G_m is denoted by μ. A function $f:G_m\to\mathbb{C}$ /$\mathbb{C}$ is the set of the complex numbers/ belongs to $L^p(G_m)$ $(1\le p<\infty)$ if and only if $|f|^p$ is integrable with respect to μ, and $\|f\|_p:=\left(\int_{G_m}|f|^p d\mu\right)^{1/p}$. $C(G_m)$ denotes the space of those complex valued functions defined on G_m, which are continuous with respect to the topology of G_m, and $\|f\|_C:=\sup_{x\in G_m}|f(x)|$ $(f\in C(G_m))$.

The elements of G_m can be represented in the form $x=(x_0,x_1,\ldots,x_k,\ldots)$ $(0\le x_k<m_k,\ k\in\mathbb{N})$. Let $I_n(x):=\{y\in G_m \mid y_k=x_k,\ k<n\}$ $(n\in\mathbb{N}\setminus\{0\})$.

It is clear that every $n\in\mathbb{N}$ can be written in the form $n=\sum_{k=0}^{\infty} n_k M_k$, where $M_0:=1$ and $M_n:=\prod_{k=1}^{n-1} m_k$ $(n\in\mathbb{N}\setminus\{0\})$. Denote $\hat{G}_m=\{\psi_n \mid n\in\mathbb{N}\}$ the character system of G_m ordered in the Walsh--Paley sense, i.e.

$$\psi_n:=\prod_{k=1}^{\infty} r_k^{n_k}, \text{ where } r_k(x):=\exp\frac{2\pi i x_k}{m_k} \quad (k\in\mathbb{N},\ x\in G_m,\ i:=\sqrt{-1}).$$

These systems are termed Vilenkin-systems. $D_n:=\sum_{k=0}^{n-1}\psi_k$ $(n\in\mathbb{N})$ are the so called Dirichlet kernels. If $f\in L(G_m)$, then $\hat{f}(k):=\int_{G_m} f\,\overline{\psi}_k\,d\mu$ $(k\in\mathbb{N})$ is the k-th Vilenkin-Fourier coefficient of f, and $S_n f:=\sum_{k=0}^{n-1}\hat{f}(k)\psi_k$ is the n-th partial sum of Vilenkin-Fourier series of f. Let $f^*:=\sup_n |S_{M_n} f|$ $(f\in L(G_m))$. We say that f belongs to the Hardy-space $H(G_m)$ if and only if $f^*\in L(G_m)$, and let $\|f\|_H:=\|f^*\|_1$.

We shall use the common notation $Y(G_m)$ for the spaces $L^1(G_m)$, $C(G_m)$, $H(G_m)$ and the natation $X(G_m)$ for the spaces $L^p(G_m)$ $(1 \leq p < \infty)$, $C(G_m)$, $H(G_m)$. The following concept of the derivative due to C. W. Onneweer [3]:

The function $f \in X(G_m)$ has a strong derivative $df \in X(G_m)$ if $\lim_n \| df - d_n f \|_X = 0$, where

$$d_n f\, x := \sum_{j=0}^{n-1} M_j^{-1} \sum_{k=0}^{m_j-1} k m_j^{-1} \sum_{\ell=0}^{m_j-1} \overline{r_j(\ell e_j)}^k f(x \dotplus \ell e_j) \quad (x \in G_m,\ n \in \mathbb{N}\setminus\{0\},$$

$e_j := (0,\dots,0,\overset{j}{1},0,\dots) \in G_m$, $\ell e_j = e_j \dotplus e_j \dotplus \dots \overset{\ell}{\dotplus} e_j$, $\dotplus$ is the group-operation on G_m).

We define the set of Vilenkin polynomials of order n as follows

$$P_n := \{ f \in L(G_m) \mid \hat{f}(k) = 0,\ k > n \} \qquad (n \in \mathbb{N}\setminus\{0\}).$$

We deal with the analogous of the classical Berstein inequalities, namely we prove the next theorems in this paper.

<u>Theorem 1</u>. Let G_m be an arbitrary Vilenkin-group $1<p<\infty$. Then

$$\| dp_n \|_p < Cn \| p_n \|_p \qquad (p_n \in P_n,\ n \in \mathbb{N}\setminus\{0\}).$$

<u>Theorem 2</u>. i/ Let G_m be an arbitrary Vilenkin group. Then for all $p_n \in P_n$ $(n \in \mathbb{N}\setminus\{0\})$

$$\| dp_n \|_Y < C \log n\, n \| p_n \|_Y .$$

ii/ If $\limsup m < \infty$, then

$$\| dp_n \|_Y < Cn \| p_n \|_Y \qquad (p_n \in P_n,\ n \in \mathbb{N}\setminus\{0\}).$$

iii/ If $\limsup m = \infty$, then ii/ fails to hold.

iv/ There exist a Vilenkin-group for which

$$\limsup \left(\sup_{p_n \in P_n} \frac{\| dp_n \|_Y}{n \log n \| p_n \|_Y} \right) > 0 .$$

C denotes an absolute positive, but not always the same constant throughout this paper. $\mathbb{0}$ denotes the zero element of G_m.

<u>Proof of Theorem 1</u>. It is easy to see that $dp_n = np_n - \sum_{k=1}^{n} S_k p_n$ $(p_n \in P_n,\ n \in \mathbb{N}\setminus\{0\})$. Since the Vilenkin system is basis in $L^p(G_m)$ $(1<p<\infty)$ [4], therefore the operators S_k $(k \in \mathbb{N}\setminus\{0\})$ are uniformly bounded from $L^p(G_m)$ to $L^p(G_m)$. From this follows that $\| dp_n \|_p < Cn \| p_n \|_p$ $(1<p<\infty)$.

Proof of Theorem 2. i/ Since $Y(G_m)$ is homogeneous Banach space [2], therefore $\|d_p p_n\|_Y = \|dD_n * p_n\|_Y \le \|dD_n\|_1 \|p_n\|_Y$ $(p_n \in P_n,\ n \in \mathbb{N}\setminus\{0\})$. It is known [6], that $\|D_k\|_1 = O(\log k)$ $(k \in \mathbb{N}\setminus\{0\})$, consequently $\|dD_n\|_1 = \|nD_n - \sum_{k=1}^{n} D_k\|_1 = O(n \log n)$. $*$ stands for convolution.

ii/ The cases $Y(G_m) = L(G_m)$, $C(G_m)$ are proved in [1]. Applying that ii/ is true for $L(G_m)$ we verify ii/ for $H(G_m)$. Let $n \in \mathbb{N}$ and $M_{k(n)} \le n < M_{k(n)+1}$. Then

$$\|dp_n\|_H = \|dD_{M_{k(n)+1}} * p_n\|_H \le \|dD_{M_{k(n)+1}}\|_1 \|p_n\|_H < CM_{k(n)+1} \|D_{M_{k(n)+1}}\|_1 \|p_n\|_H$$

$\|D_{M_k}\|_1 = 1$ $(k \in \mathbb{N})$ [6] and $M_{k(n)+1} = O(n)$ complete the proof of ii/.

iii/ For the case $L(G_m)$ it is enough to see the Vilenkin polynomials D_{M_n} $(n \in \mathbb{N})$. It is showed in [5], that

$$\|dD_{M_n}\|_1 > \int_{I_{n-1}(0)\setminus I_n(0)} |dD_{M_n}|\, d\mu > CM_n \log m_{n-1} \quad (n \in \mathbb{N}),$$

namely iii/ is true for $L(G_m)$. It is clear that $\|r_n D_{M_n}\|_H = \|D_{M_n}\|_1 = 1$ $(n \in \mathbb{N})$, and not hard to check that $(d(r_n D_{M_n}))(x) = (dD_{M_n})(x)$ $(x \in I_{n-1}(0)\setminus I_n(0))$. Obviously $r_n D_{M_n} \in P_{2M_n}$. Since $\|f\|_H \ge \|f\|_1$ $(f \in H(G_m))$, therefore iii/ is valid for $H(G_m)$. Let

$$t_{M_n} x := \begin{cases} \dfrac{\sum_{k=1}^{m_j-1} k\,(r_j(\ell e_j))^k}{\left|\sum_{k=1}^{m_j-1} k\,(r_j(\ell e_j))^k\right|} & \text{if } x \in I_n(\ell e_j)\ (0 \le j < n,\ 0 \le \ell < m_j) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $p_{M_n} \in P_{M_n}$ and $\|p_{M_n}\|_C = 1$. By elementary computation we have

$$|dp_{M_n} 0| = \left|\sum_{j=0}^{n-1} M_j m_j^{-1} \sum_{\ell=0}^{m_j-1} p_{M_n}(\ell e_j) \sum_{k=1}^{m_j-1} k\,\overline{r_j(\ell e_j)}^{\,k}\right| =$$

$$= \sum_{j=0}^{n-1} M_j m_j^{-1} \sum_{\ell=0}^{m_j-1} \left|\sum_{k=1}^{m_j-1} k\,(r_j(\ell e_j))^k\right| > CM_{n-1} m_{n-1}^{-1} \sum_{\ell=1}^{m_{n-1}-1} m_{n-1}^2 \frac{1}{\ell} >$$

$$> CM_n \log m_{n-1} \quad (n \in \mathbb{N}\setminus\{0\}).$$

iii/ is proved.

iv/ Define the sequence m inductively as follows $m_0 := 2$, $m_n := 2^{M_n}$ $(n \in \mathbb{N}\setminus\{0\})$. Thus $\log M_n = O(\log m_{n-1})$ $(n \in \mathbb{N}\setminus\{0\})$. Let

$$q_{2M_n} := \begin{cases} D_{M_n} & \text{if } Y(G_m) = L(G_m) \\ r_n D_{M_n} & \text{if } Y(G_m) = H(G_m) \\ t_{M_n} & \text{if } Y(G_m) = C(G_m) \quad (n \in \mathbb{N}). \end{cases}$$

Applying the estimations verified above we obtain that

$$\frac{\|dq_{2M_n}\|_Y}{M_n \log M_n \|q_{2M_n}\|_Y} > C \qquad (n \in \mathbb{N}) .$$

Theorem 2 is proved.

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