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BEST APPROXIMATION IN NORMED ALMOST LINEAR SPACES

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In [3], [4] we introduced and studied the normed almost linear space (nals) and strong normed almost linear space (snals), both generalizing the normed linear space (nls), and we showed ([3]) that the theory of best simultaneous approximation in a nls (or, in another terminology, the theory of Chebyshev centers) is a particular case of the theory of best approximation in a nals (snals).

In a nals (snals) one can try to extend results from the theory of best (or best simultaneous) approximation in a nls, and in [3] we began such a study. In the present paper we continue this study, to support the idea that the nals and snals constitute the natural framework for the theory of best simultaneous approximation.

All spaces involved in this paper are over the real field R.

We recall some definitions, notation and examples of [3], necessary for the understanding of this paper.

An almost linear space (als) is a set X together with two mappings $s:X\times X\longrightarrow X$ and $m:R\times X\longrightarrow X$ satisfying the conditions L_1-L_8 given below. For $x,y\in X$ and $\lambda\in R$ we denote $s(x,y)$ by $x+y$ and $m(\lambda,x)$ by λx. Let $x,y,z\in X$ and $\lambda,\mu\in R$. L_1) $(x+y)+z=x+(y+z)$; L_2) $x+y=y+x$; L_3) There exists an element $0\in X$ such that $x+0=x$; L_4) $1x=x$; L_5) $\lambda(x+y)=\lambda x+\lambda y$; L_6) $0x=0$; L_7) $\lambda(\mu x)=(\lambda\mu)x$; L_8) $(\lambda+\mu)x=\lambda x+\mu x$ for $\lambda\geqslant 0,\mu\geqslant 0$. We denote $-1x$ by $-x$, and in the sequel $x-y$ means $x+(-y)$.

Let $V_X=\{x\in X: x-x=0\}$. By L_1-L_8 it follows that V_X is a linear space.

A norm on the als X is a functional $\|\cdot\|:X\longrightarrow R$ satisfying N_1-N_4 below. Let $x,y,z\in X$ and $\lambda\in R$. N_1) $\|x-z\|\leq\|x-y\|+\|y-z\|$; N_2) $\|\lambda x\|=|\lambda|\,\|x\|$; N_3) $\|x\|=0$ iff $x=0$; since $(V_X,\|\cdot\|)$ is a nls, the weak convergence (denoted by $\rightharpoonup$) can be defined in V_X.

N_4) If $\{v_n\}_{n\in\Delta}$ is a net in V_X, $v \in V_X$, $v_n \longrightarrow v$, then for each $x \in X$, $\|x-v\| \leq \liminf \|x-v_n\|$. Clearly, $\|x\| \geq 0$ for each $x \in X$. An als X together with $\|\cdot\|: X \longrightarrow R$ satisfying N_1-N_4 is called a <u>normed almost linear space</u> (nals). For $x \in X$ and $r > 0$ let $B_X(x,r) = \{y \in X; \|y-x\| \leq r\}$. Then $B_X(x,r)$ is a convex (possibly empty) subset of X.

Since the norm of a nals X does not generate a metric on X, we shall sometimes work in the class of strong normed almost linear spaces. A <u>strong normed almost linear space</u> (snals) is a nals X together with a semi-metric ρ on X which satisfies M_1-M_3 below. Let $x,y,z \in X$ and $\lambda \in R$. M_1) $|\,\|x\| - \|y\|\,| \leq \rho(x,y) \leq \|x-y\|$; M_2) $\rho(x+z,y+z) \leq \rho(x,y)$; M_3) The function $\lambda \longrightarrow \rho(\lambda x,x)$ is continuous at $\lambda = 1$. As we have observed in [3], if X is a nls then the only semi-metric on X satisfying M_1-M_3 is that generated by the norm.

Let X be a nals and let $\emptyset \neq G \subset V_X$. We define $R_X(G) \subset X$ in the following way: $x \in R_X(G)$ if for each $g \in G$ there exists $v_g \in V_X$ such that $\|x-g\| = \|v_g-g\|$ and $\|x-v\| \geq \|v_g-v\|$ for each $v \in V_X$.

We give now examples of (strong) normed almost linear spaces which are related with the subject matter of this paper. For other examples see [3],[4].

<u>Example</u> 1. Let $(E,\|\cdot\|)$ be a nls and let X be the collection of all nonempty, bounded and convex subsets A of E. Define $s(A_1,A_2) = A_1+A_2$ and $m(\lambda,A) = \lambda A$. Let $0 \in X$ be the set $\{0\}$. Then X is an als and we have $V_X = \{\{x\} : x \in E\} \equiv E$. For $A \in X$, let $|||A||| = \sup_{a\in A} \|a\|$. Then X together with $|||\cdot|||$ is a nals. It is a snals for the Hausdorff semi-metric defined by

$$(1)\quad \rho(A_1,A_2) = \max\left\{\sup_{a_1\in A_1}\inf_{a_2\in A_2}\|a_1-a_2\|,\ \sup_{a_2\in A_2}\inf_{a_1\in A_1}\|a_1-a_2\|\right\}$$

Let G be a nonempty subset of V_X. If A is remotal with respect to G (i.e., for each $g \in G$ there exists $a_g \in A$ such that $\sup_{a\in A}\|a-g\| = \|a_g-g\|$) then $A \in R_X(G)$. The converse is not always true ([3]).

<u>Example</u> 2. Let $(E,\|\cdot\|)$ be a nls and let X be the collection of all nonempty, bounded, closed and convex subsets A of E. Define $s(A_1,A_2) = \overline{A_1+A_2}$, and define m and $0 \in X$ as above. Then X is an als. Endowed with the same norm as in Example 1, it is a nals, and V_X, $R_X(G)$ have a similar description. Together with ρ defined by (1) it is a snals. Notice that now ρ is a metric on X.

<u>Example</u> 3. Let $(E,\|\cdot\|)$ be a nls and let X be the collection of all compact, convex, nonempty subsets of E. Let $s,m,0,|||\cdot|||$ and ρ be

defined as in Example 2. Then X is a snals. For each nonempty $G \subset V_X$ we have $R_X(G) = X$.

If G is a subset of the nals X and $x \in X$, we define

(2) $\mathrm{dist}(x,G) = \inf_{g \in G} \|x-g\|$

(3) $P_G(x) = \{ g_o \in G: \|x-g_o\| = \mathrm{dist}(x,G) \}$

and we keep the same definitions for proximinal and Chebyshev sets as in a nls. As we have observed in [3] , the theory of best simultaneous approximation in a nls is a particular case of the theory of best approximation in a nals X by elements of subsets $G \subset V_X$. Indeed, in Example 1 if $x \in X$ stands for the bounded, convex, nonempty set $A \subset E$, then for any $G \subset V_X$ $(\equiv E)$ we have $\mathrm{dist}(x,G) = \mathrm{rad}_G(A)$ and $P_G(x) = \mathrm{cent}_G(A)$, where $\mathrm{rad}_G(A)$ is the Chebyshev radius of A with respect to G and $\mathrm{cent}_G(A)$ is the set of all best simultaneous approximations (or Chebyshev centers) of A with respect to G.

We give now some results concerning the sets $P_G(x)$ when $G \subset V_X$.

PROPOSITION 1. *Let G be a one-dimensional Chebyshev subspace of V_X. Then $P_G(x)$ is a singleton for each $x \in R_X(G)$.*

Proof. Clearly, G is proximinal in X. Let now $x \in R_X(G)$ and suppose there exist $g_1,g_2 \in P_G(x)$, $g_1 \neq g_2$. For $(g_1+g_2)/2 \in G$ let $v_o \in V_X$ such that

$$\| x - \frac{g_1+g_2}{2} \| = \| v_o - \frac{g_1+g_2}{2} \|$$

$$\| x-v \| \geq \| v_o-v \| \quad \text{for each } v \in V_X$$

Since $(g_1+g_2)/2 \in P_G(x)$, it follows that

$$\mathrm{dist}(x,G) = \| v_o - \frac{g_1+g_2}{2} \| \leq \frac{\|v_o-g_1\|+\|v_o-g_2\|}{2} \leq \frac{\|x-g_1\|+\|x-g_2\|}{2} = \mathrm{dist}(x,G)$$

and so $\| v_o-(g_1+g_2)/2 \| = \| v_o-g_i \|$, i=1,2. Since dim G = 1, we must have $g_1,g_2 \in P_G(v_o)$, a contradiction. Therefore $g_1=g_2$, i.e., $P_G(x)$ is a singleton.

An immediate consequence of this result is the following application for best simultaneous approximation.

COROLLARY 1. *Let G be a one-dimensional Chebyshev subspace of the nls E. Then $\mathrm{cent}_G(A)$ is a singleton for each nonempty, bounded subset $A \subset E$, when A is remotal with respect to G.*

In ([1], Theorem 2) it was proved that if the norm of a Banach space is uniformly Kadec-Klee (UKK), then Chebyshev centers with respect to w-compact convex sets are compact. We recall ([5]) that the norm of a Banach space E is (UKK) if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the relations $x_n \in E$, $\|x_n\| \leq 1$, $n=1,2,\dots$, $x_n \longrightarrow x$, $\operatorname{sep}\{x_n\} = \inf\{\|x_n - x_m\| : m \neq n\} \geq \varepsilon$ imply that $\|x\| \leq 1-\delta$. For a nals X we have the following result:

THEOREM 1. <u>Let</u> X <u>be a nals such that</u> V_X <u>is a Banach space and the norm of</u> V_X <u>is</u> (UKK), <u>and let</u> $G \subset V_X$ <u>be a</u> w-<u>compact, convex set. Then for each</u> $x \in R_X(G)$ <u>the set</u> $P_G(x)$ <u>is compact (and convex).</u>

<u>Proof</u>. Clearly G is proximinal in X. Let now $x \in R_X(G)$. If $P_G(x)$ is not compact there exists a sequence $\{g_n\} \subset P_G(x)$ with $\operatorname{sep}\{g_n\} \geq \varepsilon$ for some $\varepsilon > 0$. Since $P_G(x)$ is w-compact, we may assume (passing to a subsequence) that $g_n \longrightarrow g \in P_G(x)$. Since $x \in R_X(G)$, for $g \in G$ there exists $v_g \in V_X$ such that $\|x-g\| = \|v_g - g\|$ and $\|x-g_n\| \geq \|v_g - g_n\|$, $n=1,2,\dots$. Hence $r=\|g-v_g\| \geq \sup_{n\in N}\|g_n - v_g\|$. Choose $\delta = \delta(\varepsilon/r) > 0$ as in the definition of (UKK). Then $r^{-1}\|g_n - v_g\| \leq 1$, $r^{-1}(g_n - v_g) \longrightarrow r^{-1}(g-v_g)$ and $\operatorname{sep}\{r^{-1}(g_n-v_g)\} \geq r^{-1}\varepsilon$, whence by (UKK) we obtain that $r^{-1}\|g-v_g\| \leq 1-\delta$, a contradiction.

<u>Remarks</u>. a) An immediate proof of Theorem 1 can be obtained using Theorem 4.8 of [3] and Theorem 2 of [1] . It is so no surprising that we have used in the above proof some ideas of the proof of Theorem 2 of [1] . b) When applied to best simultaneous approximation Theorem 1 is weaker than the corresponding result of [1] . Is Theorem 1 true for any $x \in X$?

Mach ([7]) introduced a certain property of a pair (E,G), where E is a nls and G a nonempty subset of E. This property can be formulated in a similar way for a pair (X,G), where X is a nals and $G \subset V_X$, and below we give the definition in this more general framework.

DEFINITION 1. Let X be a nals and $\emptyset \neq G \subset V_X$. The pair (X,G) is said to have the <u>property</u> (P) if for every $r > 0$ and every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ and a function $h: G \times G \longrightarrow G$ such that for every $|\theta| < \delta(\varepsilon)$ we have $h(g_1,g_2) \in B_X(g_1,\varepsilon)$ and $B_X(g_1, r+\delta(\varepsilon)) \cap B_X(g_2, r+\theta) \subset B_X(h(g_1,g_2), r+\theta)$.

Examples of normed linear spaces E and $G \subset E$ such that the pair (E,G) has the property (P) can be found in [7] . Here we give an example of a nals X and $G \subset V_X$ such that the pair (X,G) has the property (P).

Example 4. Let E be a nls and $G \subset E$ such that the pair (E,G) has the property (P). Let X be the nals corresponding to E, given in one of the Examples 1,2 or 3. Then $G \subset V_X$ and we show that the pair (X,G) has the property (P). Let $r > 0$ and $\varepsilon > 0$ be given, and let $\delta(\varepsilon)$ and $h: G \times G \longrightarrow G$ be given by the property (P) of the pair (E,G). Then $h(g_1,g_2) \in B_X(g_1,\varepsilon)$. Let $|\theta| < \delta(\varepsilon)$ and $A \in X$ such that $A \in B_X(g_1, r+\delta(\varepsilon)) \cap B_X(g_2, r+\theta)$. Then for each $a \in A$ we have $\|a-g_1\| \leq |||A-g_1||| \leq r+\delta(\varepsilon)$ and $\|a-g_2\| \leq |||A-g_2||| \leq r+\theta$. Since (E,G) has the property (P), for each $a \in A$ we have $\|a-h(g_1,g_2)\| \leq r+\theta$, and so $|||A-h(g_1,g_2)||| \leq r+\theta$, i.e., (X,G) has the property (P).

The next result generalizes for a nals ([7], Theorem 2) given for best simultaneous approximations. The proof is omitted, being essentially the same with that of ([7]).

THEOREM 2. Let X be a nals and G a complete subset of V_X. If the pair (X,G) has the property (P), then G is proximinal in X.

Let now X be a snals and $G \subset X$. Then the semi-metric ρ generates a topology on X and so we can discuss the continuity (semi-continuity) properties of the set-valued mapping $x \longrightarrow P_G(x)$. In the sequel we need some facts given in [3], which we collect them in a lemma.

LEMMA 1. Let X be a snals and $G \subset X$.

i) If $x,y \in X$, then $|\mathrm{dist}(x,G)-\mathrm{dist}(y,G)| \leq \rho(x,y)$.

ii) If $x \in X$, $y \in \mathrm{Dom}(P_G) (= \{x \in X : P_G(x) \neq \emptyset\})$ and $g \in P_G(y)$, then $\|x-g\| \leq \mathrm{dist}(x,G)+2\rho(x,y)$.

iii) If $G \subset V_X$ then P_G is both upper and lower semicontinuous at any $g \in G$.

Generalizing ([8], Definition, p.227) for a nals X, we give the following definition.

DEFINITION 2. Let X be a nals, $G \subset X$ and $A \subset \mathrm{Dom}(P_G)$. The pair (G,A) is said to have the property (P_1) if for every $x \in A$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $g \in G$ with $\|x-g\| < \mathrm{dist}(x,G)+\delta$ we have that $\mathrm{dist}(g,P_G(x)) < \varepsilon$. The pair (G,A) is said to have the property (P_2) if it has property (P_1) such that $\delta > 0$ can be chosen independently of $x \in A$.

The next result extends ([8], Theorems 5 and 6) from the theory of best simultaneous approximation.

THEOREM 3. Let X be a snals, $G \subset X$ and $A \subset Dom(P_G)$. If (G,A) has the property (P_1) then $P_G | Dom(P_G)$ is upper Hausdorff semicontinuous at any $x \in A$. If (G,A) has the property (P_2) then $P_G | A$ is Hausdorff continuous on A.

Proof. Let $x \in A$ and $x_n \in Dom(P_G)$ such that $\rho(x_n,x) \to 0$. Let $\varepsilon > 0$ and choose $\delta > 0$ given by the property (P_1) for the element $x \in A$. Let N_ε be such that $\rho(x_n,x) < \delta/2$ for $n > N_\varepsilon$. Let $g_n \in P_G(x_n)$. Then by Lemma 1, ii) we have $\|x-g_n\| < dist(x,G)+\delta$ for $n > N_\varepsilon$. By the property (P_1) it follows that $dist(g_n,P_G(x)) < \varepsilon$ for $n > N_\varepsilon$, and so $\sup\{dist(g_n,P_G(x)) : g_n \in P_G(x_n)\} \le \varepsilon$ for $n > N_\varepsilon$, i.e., P_G is upper Hausdorff semicontinuous at x.

Suppose (G,A) has the property (P_2). By the above we must show only that $P_G | A$ is lower Hausdorff semicontinuous on A. Let $x_n, x \in A$ be such that $\rho(x_n,x) \to 0$. Let $\varepsilon > 0$ and choose $\delta > 0$ independently of $x \in A$ given by the property (P_2). We have $\rho(x_n,x) < \delta/2$ for $n > N_\varepsilon$. Let $g_o \in P_G(x)$. By Lemma 1, ii) we get $\|x_n-g_o\| < dist(x_n,G)+\delta$ for $n > N_\varepsilon$, and so by property (P_2) we have that $dist(g_o,P_G(x_n)) < \varepsilon$ for $n > N_\varepsilon$, whence $\sup\{dist(g_o,P_G(x_n)) : g_o \in P_G(x)\} \le \varepsilon$ for $n > N_\varepsilon$, which completes the proof.

Using property (P) we have the following sufficient condition for P_G to be lower semicontinuous. It generalizes ([7], Theorem 3) given for best simultaneous approximation.

THEOREM 4. Let X be a snals and G a complete subset of V_X such that the pair (X,G) has the property (P). Then P_G is lower semicontinuous on X.

Proof. By Theorem 2, G is proximinal in X. Let now $x \in X$. By Lemma 1, iii) we can suppose $x \notin G$ and so $r = dist(x,G) > 0$. Let $g_o \in P_G(x)$, $\varepsilon > 0$ and let $\delta(\varepsilon)$ be given by the property (P). We show that for each $y \in X$, $\rho(y,x) < \delta(\varepsilon)$ we have that $P_G(y) \cap \{g \in G : \rho(g,g_o) < 2\varepsilon\} \neq \emptyset$ which will prove that P_G is lower semicontinuous at x. Let $\theta = \rho(y,x) < \delta(\varepsilon)$. By Lemma 1, i) we have $|dist(y,G)-dist(x,G)| = |dist(y,G)-r| \le \rho(y,x) = \theta$ and so $dist(y,G) = r+\alpha$ for some $\alpha \in R$, $|\alpha| \le \theta < \delta(\varepsilon)$. Let $g \in P_G(y)$. Then $\|y-g\| = dist(y,G) = r+\alpha$ and since $\|y-g_o\| = \rho(y,g_o) \le \rho(y,x)+\rho(x,g_o) = \rho(y,x)+dist(x,G) < \delta(\varepsilon)+r$, it follows that $y \in B_X(g_o,r+\delta(\varepsilon)) \cap B_X(g,r+\alpha)$. By the property (P) we have for $g_1 = h(g_o,g)$ that $\|y-g_1\| \le r+\alpha = dist(y,G)$, i.e., $g_1 \in P_G(y)$ and $\rho(g_1,g_o) = \|g_1-g_o\| \le \varepsilon < 2\varepsilon$.

Up to now we gave sufficient conditions to ensure some semi-continuity properties of $P_G: X \longrightarrow 2^G$ at some $x \in X$. We conclude this paper with a caracterization of lower (upper) semicontinuity of P_G when G is a linear subspace of V_X which is proximinal in X. The first resul generalizes [6] and the second one [2], where they were stated for a nls. The proofs are omitted, being similar with those from ([6],[2]).

We recall the following definition of [9] , which we formulate it in a nals.

DEFINITION 3. Let X be a nals and $G \subset V_X$ a linear subspace which is proximinal in X. A selection $s: X \longrightarrow G$ for P_G is said to have the "Nulleigenschaft", if $s(x) = 0$ for each $x \in P_G^{-1}(0)$.

PROPOSITION 2. Let X be a snals such that ρ is a metric on X. Let G be a complete linear subspace of V_X which is proximinal in X. Then P_G is lower semicontinuous on X iff P_G admits a continuous selection with the "Nulleigenschaft".

THEOREM 5. Let X be a snals such that ρ is a metric on X. Let G be a linear subspace of V_X which is proximinal in X. Then P_G is upper semicontinuous on X iff for each compact subset $A \subset X$, the set $P_G(A) = \bigcup \{ P_G(x) : x \in A \}$ is compact.

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