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ON APPROXIMATION IN C(X)

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1. Introduction. The present paper extends some of the author's recent results on approximation of certain operators A by positive linear operators both being defined on C(X). Here C(X) = C(X,d) is the space of real-valued continuous functions on a compact metric space (X,d) with diameter d(X) > 0. See [3] for details. As was the case in this paper, the estimates will be obtained by applying the smoothing technique, based upon the use of

$$K(t,f;C(X),\mathrm{Lip}\,1):=\inf\{\|f-g\|+t\cdot|g|_{\mathrm{Lip}\,1}: g\in \mathrm{Lip}\,1\}.$$

Here $t\geq 0$, $f\in C(X)$, $\|f\|:=\sup\{|f(x)|: x\in X\}$, and

$$\mathrm{Lip}\,1:=\{g\in C(X): |g|_{\mathrm{Lip}\,1}:=\sup\{|g(x)-g(y)|/d(x,y): d(x,y)>0\}<\infty\}.$$

A lemma of Yu. A. Brudnyĭ shows that for $\varepsilon\geq 0$ one has

$$K(\tfrac{1}{2}\varepsilon,f;C(X),\mathrm{Lip}\,1)=\tfrac{1}{2}\cdot\tilde{\omega}(f,\varepsilon),$$

where $\tilde{\omega}(f,\cdot)$ denotes the least concave majorant of the modulus of continuity $\omega(f,\cdot)$, given for $\varepsilon>0$ by $\omega(f,\varepsilon)=\sup\{|f(x)-f(y)|: d(x,y)\leq\varepsilon\}$.

As shown in [3], for a space (X,d) having a so-called coefficient of convex deformation $\rho=\rho(X)\geq 1$, $\tilde{\omega}$ and ω are related by

$$\omega(f,\varepsilon)\leq\tilde{\omega}(f,\varepsilon)\leq(1+\rho)\cdot\omega(f,\varepsilon).$$

Thus for these spaces our below estimates given in terms of $\tilde{\omega}$ imply such in terms of ω. However, the assumption that (X,d) has a coefficient of convex deformation is quite restrictive, showing that the approach via $\tilde{\omega}$ is the more general one.

For the direct approach the reader is referred to recent papers by M.A. Jiménez Pozo [5] and T. Nishishiraho [7,8,9]; see the references in these papers for what had been done on the subject earlier.

2. Estimates on Approximation by Bounded Linear Operators. The underlying smoothing technique is described in

Theorem 2.1 Let $Y\neq\emptyset$ be some set and let B(Y) denote the space of real-valued and bounded functions on Y. If Δ is a bounded linear operator mapping C(X) into B(Y) such that for some $y \in Y$ one has

$$|\Delta(g,y)| \le \varphi(y) \cdot |g|_{Lip\,1} \text{ for } \varphi(y) \ge 0 \text{ and all } g \in Lip\,1,$$

then for all $f \in C(X)$ and $\varepsilon > 0$ inequality

$$|\Delta(f,y)| \le \max\{\tfrac{1}{2}\|\Delta\|, \varphi(y) \cdot \varepsilon^{-1}\} \cdot \tilde{\omega}(f,\varepsilon)$$

holds.

Proof. The inequality is obviously correct for $\Delta = 0$, so let $\Delta \ne 0$. Let g be arbitrarily given in Lip 1. For any $f \in C(X)$ we have

$$|\Delta(f,y)| \le |\Delta(f-g,y)| + |\Delta(g,y)|$$
$$\le \|\Delta\| \cdot \|f-g\| + \varphi(y) \cdot |g|_{Lip\,1}$$
$$= \|\Delta\| \cdot (\|f-g\| + \|\Delta\|^{-1} \cdot \varphi(y) \cdot |g|_{Lip\,1})$$

Passing to the inf in Lip 1 implies for each $\varepsilon > 0$

$$|\Delta(f,y)| \le \|\Delta\| \cdot K(\|\Delta\|^{-1} \cdot \varphi(y) \cdot \tfrac{1}{2}\varepsilon \cdot (\tfrac{1}{2}\varepsilon)^{-1}, f; C(X), Lip\,1)$$
$$\le \|\Delta\| \cdot \max\{1, \|\Delta\|^{-1} \cdot \varphi(y) \cdot (\tfrac{1}{2}\varepsilon)^{-1}\} \cdot K(\tfrac{1}{2}\varepsilon, f; C(X), Lip\,1)$$
$$= \max\{\|\Delta\|, \varphi(y) \cdot (\tfrac{1}{2}\varepsilon)^{-1}\} \cdot K(\tfrac{1}{2}\varepsilon, f; C(X), Lip\,1).$$

Brudnyi's lemma implies

$$|\Delta(f,y)| \le \max\{\tfrac{1}{2}\|\Delta\|, \varphi(y) \cdot \varepsilon^{-1}\} \cdot \tilde{\omega}(f,\varepsilon). \quad \square$$

The next theorem deals with approximation of operators A given for $f \in C(X)$ and $y \in Y$ by $A(f,y) = \Psi_A(y) \cdot f(g_A(y))$ where $\Psi_A \in B(Y)$ and g_A maps Y into X.

Theorem 2.2 Let A be given as above and L be a bounded linear operator, both mapping C(X) into B(Y). Then for $f \in C(X)$, $y \in Y$, and $0 < \varepsilon$ we have

$$|(L-A)(f,y)| \le$$
$$\max\{\tfrac{1}{2}(\|L\| + \|L(1_X)\|), \varepsilon^{-1}[d(X)(\|\varepsilon_y \circ L\| - L(1_X,y)) + |L(d(\cdot,g_A(y));y)|]\} \tilde{\omega}(f,\varepsilon)$$
$$+ |(L-A)(1_X,y)| \cdot |f(g_A(y))|.$$

Here 1_X denotes the function $X \ni x \mapsto 1 \in \mathbb{R}$.

The proof of Theorem 2.2 is obtained by using Theorem 2.1 and a suitable estimate for Lipschitz functions g. $\square$

Remark 2.3 For estimates similar to the one of Theorem 2.2 using $\omega(f,\cdot)$

instead of $\tilde{\omega}(f,\cdot)$ see e.g. M.A. Jiménez Pozo's paper [4].

3. Positive Linear Operators. For these Theorem 2.2 implies the following

Corollary 3.1 (see [3, Th. 3.1]). Let A be given as above, and L be a positive linear operator, both mapping C(X) into B(Y). Then the following inequality holds for $f \in C(X)$, $y \in Y$, and $\varepsilon > 0$

$$|(L-A)(f,y)| \le \max\{\|L\|, \varepsilon^{-1} \cdot L(d(\,,g_A(y));y)\} \cdot \tilde{\omega}(f,\varepsilon) + |(L-A)(1_X,y)| \cdot |f(g_A(y))|.$$

As was observed by several authors earlier (see e.g. the papers of Jiménez Pozo and Nishishiraho), both quantities $(L-A)(1_X,y)$ and $L(d(\,,g_A(y));y)$ can be replaced by similar expressions which may be easier to evaluate We briefly give an example of what can be done in this direction for the positive linear operator case; proofs of these and related estimates will be presented elsewhere.

Theorem 3.2 Let A, L, f, y, ε be given as in Corollary 3.1, and let $h \in C(X)$ be such that $f/h \in C(X)$.

If $\Phi : X^2 \to \mathbb{R}$ is such that $\Phi(\cdot,y) \in C(X)$ for all $y \in X$, and that for some $q \ge 1$ the condition $d(x,y)^q \le \Phi(x,y)$, $x, y \in X$, holds, then

$$|(L-A)(f,y)| \le \max\{\tfrac{1}{2}(\|L\|\,\|h\| + |L(h,y)|)\,,\ \varepsilon^{-1}\ \|h\|\ \ C(L,\Phi,A,y)\} \cdot \tilde{\omega}(f/h,\varepsilon)$$
$$+ |(f/h)(g_A(y))|\ \ |(L-A)(h,y)|\,,$$

where

$$C(L,\Phi,A,y) := \inf\{L(\Phi(\,,g_A(y))^{p/q};y)^{1/p}\ \ L(1_X;y)^{1-1/p} : p \ge 1\}.$$

Corollary 3.3 For the special case $Y = X$, $A(f,x) = f(x)$, $h = 1_X$, $L1_X = 1_X$, we have

$$|L(f,x) - f(x)| \le \max[1, \varepsilon^{-1}\ \inf\{L(\Phi(\cdot,x)^{p/q};x)^{1/p} : p \ge 1\}] \cdot \tilde{\omega}(f,\varepsilon).$$

Further generalizations of the basic inequality of Corollary 3.1 may be obtained following the lines in the author's paper [3].

For estimates similar to the ones of this section using $\omega(f,\cdot)$ instead of $\tilde{\omega}(f,\cdot)$ see for example the articles of T. Nishishiraho.

4. Applications. There are a wide variety of applications of the above results to approximation of multivariate functions by both positive and non-positive linear methods. Here we restrict ourselves to two types.

We consider first products of parametric extensions of univariate operators. This is a common way of constructing operators for the multivariate case. Following E.W. Cheney - W.J. Gordon [2] the method can be described as follows.

Let X and Y be compact spaces. Define $\hat{x}: C(X) \to \mathbb{R}$ by $\hat{x}(f) = f(x)$, and $\hat{x}: C(X \times Y) \to C(Y)$ by $(\hat{x}f)(y) = f(x,y)$. Similarly $\hat{y}: C(Y) \to \mathbb{R}$ is defined by $\hat{y}(f) = f(y)$, and $\hat{y}: C(X \times Y) \to C(X)$ by $(\hat{y}f)(x) = f(x,y)$. If $L_1: C(X) \to C(X)$ is a linear operator, then $\bar{L}_1: C(X,Y) \to C(X,Y)$ given by $(\bar{L}_1 f)(x,y) = \hat{x}\, L_1\, \hat{y}\, f$ is the parametric extension of L_1. Similarly, $\bar{L}_2: C(X,Y) \to C(X,Y)$ with $(\bar{L}_2 f)(x,y) := \hat{y}\, L_2\, \hat{x}\, f$ is the parametric extension of L_2. Parametric extensions are examples of linear operators on $C(X \times Y)$ commuting with each other.

We apply Corollary 3.1 for the case $X = [a,b]$, $Y = [c,d]$ with $a < b$, $c < d$, assuming that on $X \times Y$ the metric d_1 is given by $d_1((x_1,x_2),(y_1,y_2)) := |x_1-y_1| + |x_2-y_2|$, to obtain

<u>Theorem 4.1</u> Let $L_1: C[a,b] \to C[a,b]$ and $L_2: C[c,d] \to C[c,d]$ be positive linear operators satisfying $L_1(e_0) = L_2(e_0) = 1$, where $e_0(x)=1$. Then for any $f \in C([a,b] \times [c,d])$, $(y_1,y_2) \in [a,b] \times [c,d]$, and $\varepsilon > 0$ we have

$$|(\bar{L}_1 \circ \bar{L}_2)(f; y_1,y_2) - f(y_1,y_2)|$$
$$\leq \max\,[\,1,\, \varepsilon^{-1} \cdot \{L_1(|\cdot_1 - y_1|; y_1) + L_2(|\cdot_2 - y_2|; y_2)\}]\;\; \tilde{\omega}_{d_1}(f,\varepsilon)\,;$$

here $\tilde{\omega}_{d_1}(f,\cdot)$ denotes the least concave majorant of the modulus of continuity $\omega_{d_1}(f,\cdot)$ defined with respect to the metric d_1.

<u>Example 4.2</u> An interesting example of a product of parametric extensions is given by the following version of bivariate Matsuoka-Lehnhoff operators (for the univariate case see e.g. [3]). While the estimate given in this paper was one for algebraic polynomial operators based on certain kernels then denoted by K_{3n-3}, the kernels

$$M_{4n-4}(v) = \frac{315}{2n(151n^6 + 70n^4 + 49n^2 + 45)} \cdot \left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^8$$

may serve a similar purpose (see Y. Matsuoka [6, p.14]).

If the univariate operator given by

$$J_n(f,x) = \pi^{-1} \int_{-\pi}^{\pi} f[\cos(\arccos x + v)] \; M_{4n-4}(v)\,dv$$

is used twice to define parametric extensions $\bar{J}_{n_1}$ and $\bar{J}_{n_2}$, then Theorem 4.1 implies for their product $\bar{J}_{n_1} \circ \bar{J}_{n_2}$, any $f \in C([-1,1]^2)$ and $(y_1,y_2) \in [-1,1]^2$ inequality

$$|(\bar{J}_{n_1} \circ \bar{J}_{n_2})(f;y_1,y_2) - f(y_1,y_2)| \le \sqrt{1.6}\; \tilde{\omega}_{d_1}\left(f, \Sigma_{i=1}^{2} n_i^{-1}\{\sqrt{1-y_i^2} + n_i^{-1} |y_i|\}\right). \square$$

The second method we discuss here to construct multivariate operators yields so-called pseudo-polynomial operators. Their construction is also based on univariate mappings, but the resulting operator is not necessarily positive, even if the underlying univariate ones are. For a more recent reference we mention I. Badea [1].

The special example discussed there is based upon Bernstein operators. For each $n \in \mathbb{N}$ and $f \in C([0,1]^2)$, the sequence $P_n f$ is defined by the sum

$$P_n(f;x,y) = \tfrac{1}{2} \Sigma_{i=0}^{n} \{f(x,i/n) + f(i/n,y) - f(i/n,i/n)\} \; (p_{n,i}(x) + p_{n,i}(y));$$

where $p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$, $0 \le i \le n$, $0 \le x \le 1$.

Using the continuous function $f_0 \ge 0$ given for $0 \le x,y \le 1$ by

$$f_0(x,y) = \begin{cases} (1-\alpha) & \text{if } x+y = \alpha/2n, \quad 0 \le \alpha \le 1, \\ 0 & \text{elsewhere in } [0,1]^2, \end{cases}$$

and observing that $P_n(f_0;1/2n,1/2n) = -(1-1/2n)^n < 0$, it can be seen that P_n is not a positive operator. Also $\|P_n\| \le 3$ for all $n \in \mathbb{N}$.

In order to apply Theorem 2.1 we investigate the difference $|P_n(g;x,y) - g(x,y)|$ for Lip 1 functions g. It follows from the definition of P_n that if d_1 and $|g|_{\mathrm{Lip}\,1}$ are defined as above, then

$$|P_n(g;x,y) - g(x,y)| \le \{\Sigma_{i=0}^{n} |x-i/n| \; p_{n,i}(x) + \Sigma_{i=0}^{n} |y-i/n| \; p_{n,i}(y)\} \; |g|_{\mathrm{Lip}\,1}$$
$$= (s_n(x) + s_n(y)) \; |g|_{\mathrm{Lip}\,1}$$

Here the function s_n is given by (see F. Schurer and F.W. Steutel [10])

$$s_n(x) = (2/n) \; (n-r)\binom{n}{r} x^{r+1} (1-x)^{n-r} \quad \text{with } r = [nx].$$

Hence Theorem 2.1 yields for $f \in C([0,1]^2)$ and $\varepsilon > 0$ the inequality

$$|P_n(f;x,y) - f(x,y)| \le \max\{3/2, (s_n(x) + s_n(y)) \; \varepsilon^{-1}\} \; \tilde{\omega}_{d_1}(f,\varepsilon).$$

In view of $\tilde{\omega}_{d_1}(f,\varepsilon) \le 2\,\omega_{d_1}(f,\varepsilon)$ and $s_n(x) \le \sqrt{x(1-x)/n}$, the particular choice $\varepsilon = 1/\sqrt{n} \; 2/3 \; (\sqrt{x(1-x)} + \sqrt{y(1-y)})$ implies

$$|P_n(f;x,y) - f(x,y)| \le 3 \quad \omega_{d_1}\left(f, 2/3 \quad 1/\sqrt{n} \quad \left(\sqrt{x(1-x)} + \sqrt{y(1-y)}\right)\right).$$

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