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UNIFORM APPROXIMATION ON THE WHOLE REAL LINE

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1. Introduction. Starting from the Szász operators

$$S_n(f;x) := \sum_{k=0}^{\infty} f(\tfrac{k}{n})\, p_{n,k}(x) \qquad \left(p_{n,k}(x) := e^{-nx}\frac{(nx)^k}{k!}, \quad x \geq 0\right)$$

we define the operators

$$H_n(f;x) := \frac{1}{e^{nx}+e^{-nx}} \sum_{k=0}^{\infty} \left[f(\tfrac{k}{n}) + (-1)^k \cdot f(-\tfrac{k}{n}) \right] \frac{(nx)^k}{k!}$$

$$(n = 1,2,\ldots; \quad x \in \mathbb{R} :=]-\infty,\infty[ ).$$

In [1] we outlined the idea leading to the definition of H_n, in [2] we discussed the rate of uniform approximation for the operators H_n on finite intervals of the type $[-a,a]$. In this paper we study the uniform approximation on the set $\mathbb{R} \setminus]-a,a[$, where $a > 0$.

2. Notations. Let $A, B, \beta, \delta, M, K_1, K_2, \ldots$ be non-negative numbers, $0 < \beta \leq 1$, $0 \leq \delta \leq 1$.

Let F_j $(j=1,2,3,4)$ denote the set of continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition (C_j) below (we also suppose that g is differentiable in the case of F_2 and F_3, and twice differentiable in F_4 - on the interval $]\frac{a}{2},\infty[$).

(C_1) $\quad |g(y)-g(t)| \leq A \dfrac{(y-t)^{\beta}}{t^{\beta/2}} \qquad (\frac{a}{2} < t < y < \infty);$

(C_2) $\quad |g'(t)| \leq A/\sqrt{t} \qquad (t > \frac{a}{2});$

$$(C_3)\quad \begin{cases} |g'(y)-g'(t)| \le A\dfrac{(y-t)^{\delta}}{t^{(1+\delta)/2}} & (\frac{a}{2} < t < y < \infty), \\ |g'(t)| \le B & (t > a); \end{cases}$$

$$(C_4)\qquad |g''(t)| \le \frac{A}{t} \quad (t > \tfrac{a}{2}); \qquad |g'(t)| \le B \quad (t > a).$$

Moreover for a function $g : \mathbb{R} \to \mathbb{R}$ let

$$\|g\|_a := \sup_{|t| \ge a} |g(t)|; \qquad g^-(t) := g(-t) \quad (t \in \mathbb{R}).$$

3. Theorem (T_j) /j=1,2,3,4/. Let $f : \mathbb{R} \to \mathbb{R}$ and $a > 0$. If $f, f^- \in F_j$, then $\|H_n(f) - f\|_a = O\left(n^{-\alpha_j}\right)$ $(n \to \infty)$, where

$$\alpha_1 = \frac{\beta}{2}, \qquad \alpha_2 = \frac{1}{2}, \qquad \alpha_3 = \frac{1+\delta}{2}, \qquad \alpha_4 = 1.$$

(Our proof will show that Theorems (T_3) and (T_4) are also valid for a=0.)

4. Lemmas (L1), (L2), ..., (L8). Let $x \ge 0$, $n = 1,2,\ldots$.

(L1) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right) p_{n,k}(x) = 0;$ (L2) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 p_{n,k}(x) = \frac{x}{n};$

(L3) $$\sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right|^{\gamma} p_{n,k}(x) \le \left(\frac{x}{n}\right)^{\gamma/2} \quad (0 < \gamma \le 2);$$

(L4) $\displaystyle\sum_{|\frac{k}{n}-x| \ge \frac{x}{2}} p_{n,k}(x) \le \frac{4}{nx};$ (L5) $\displaystyle\sum_{|\frac{k}{n}-x| \ge \frac{x}{2}} \left|\frac{k}{n}-x\right| p_{n,k}(x) = \frac{2}{n};$

(L6) $\displaystyle\left|\sum_{|\frac{k}{n}-x| < \frac{x}{2}} \left(\frac{k}{n}-x\right) p_{n,k}(x)\right| \le \frac{2}{n};$ (L7) $\displaystyle\left|\sum_{|\frac{k}{n}-x| < \frac{x}{2}} \left(\frac{k}{n}-x\right)(-1)^k p_{n,k}(x)\right| \le \frac{3}{n};$

(L8) $$H_n(f; -x) = H_n(f^-; x) \qquad (x \in \mathbb{R}).$$

5. The proof of the Lemmas. A simple computation gives the validity of the Lemmas (L1), (L2), and (L8). (L3) follows from (L2) using the Hölder inequality

$$\sum_k a_k b_k \le \left(\sum_k a_k\right)^{1/\alpha} \left(\sum_k b_k^{\beta}\right)^{1/\beta} \qquad (a_k, b_k > 0,\ \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1)$$

with $a_k = \left(\left|\frac{k}{n}-x\right|^2 p_{n,k}(x)\right)^{\gamma/2}$, $b_k = \left(p_{n,k}(x)\right)^{(2-\gamma)/2}$, $\alpha=\frac{2}{\gamma}$, $\beta=\frac{2}{2-\gamma}$.

(L4) follows from (L2), taking into account that

$$\frac{x}{n} = \sum_{k=0}^{\infty} \left|\frac{k}{n}-x\right|^2 p_{n,k}(x) \geq \sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{n}} \left|\frac{k}{n}-x\right|^2 p_{n,k}(x) \geq \left(\frac{x}{2}\right)^2 \sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} p_{n,k}(x).$$

From (L2) and (L4), using the Cauchy inequality we get (L5):

$$\sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} \left|\frac{k}{n}-x\right| p_{n,k}(x) \leq \left(\sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} \left|\frac{k}{n}-x\right|^2 p_{n,k}(x)\right)^{1/2} \left(\sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} p_{n,k}(x)\right)^{1/2} \leq$$

$$\leq \left(\frac{x}{n}\,\frac{4}{nx}\right)^{1/2} = \frac{2}{n}.$$

To prove (L6) we start from (L1):

$$0 = \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right) p_{n,k}(x) = \sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} \ldots + \sum_{\left|\frac{k}{n}-x\right| < \frac{x}{2}} \ldots .$$

(L6) follows from this equality and (L5):

$$\left|\sum_{\left|\frac{k}{n}-x\right| < \frac{x}{2}} \ldots\right| = \left|\sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} \ldots\right| \leq \sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} \left|\frac{k}{n}-x\right| p_{n,k}(x) \leq \frac{2}{n}.$$

A simple computation gives that $\sum_{k=0}^{\infty} \left(\frac{k}{n}-x\right)(-1)^k p_{n,k}(x) = -2xe^{-2nx}$.

Hence, by using (L5) we get

$$\sum_{\left|\frac{k}{n}-x\right| < \frac{x}{2}} \left(\frac{k}{n}-x\right)(-1)^k p_{n,k}(x) \leq \frac{2}{n} + 2xe^{-2nx} = \frac{2}{n}\left(1+nxe^{-2nx}\right) \leq \frac{3}{n}.$$

6. Preliminary part of the proof of the Theorems. It is enough to show that

(1) $\quad f, f^- \in F_j \Rightarrow \sup_{x \geq a} \left|H_n(f;x) - f(x)\right| = o\left(n^{-\alpha_j}\right) \quad (n \to \infty)$.

Indeed, if $f, f^- \in F_j$, then $f^-, \left(f^-\right)^- \in F_j$, therefore (1) implies

(2) $\quad f, f^- \in F_j \Rightarrow \sup_{x \geq a} \left|H_n(f^-;x) - f^-(x)\right| = o\left(n^{-\alpha_j}\right) \quad (n \to \infty)$.

Using (L8), it follows from (2) that

(3) $\quad f, f^- \in F_j \Rightarrow \sup_{x \geq a} \left|H_n(f;-x) - f(-x)\right| = o\left(n^{-\alpha_j}\right) \quad (n \to \infty)$.

Thus (1) implies (3). Therefore, for the rest of this paper we assume that $x \geq a$.

$$H_n(f;x) - f(x) =$$

$$= \frac{1}{e^{nx}+e^{-nx}} \left\{ \sum_{k=o}^{\infty} [f(\tfrac{k}{n}) + (-1)^k \cdot f(-\tfrac{k}{n})] \frac{(nx)^k}{k!} - f(x) \sum_{k=o}^{\infty} [1+(-1)^k] \frac{(nx)^k}{k!} \right\} =$$

$$= \frac{1}{1+e^{-2nx}} \left\{ \sum_{k=o}^{\infty} [f(\tfrac{k}{n}) - f(x)] p_{n,k}(x) + \sum_{k=o}^{\infty} [f^-(\tfrac{k}{n}) - f^-(x)](-1)^k p_{n,k}(x) + \right.$$

$$\left. + [f(-x) - f(x)] e^{-2nx} \right\} =: \frac{1}{1+e^{-2nx}} (U + V + Z).$$

7. The proof of Theorem (T_1). Let $x \geq a$, where $a > 0$.

$$|U| \leq \sum_{k=o}^{\infty} \left| f(\tfrac{k}{n}) - f(x) \right| p_{n,k}(x) = \sum_{\frac{k}{n}<\frac{x}{2}} + \sum_{\frac{k}{n}\geq\frac{3x}{2}} + \sum_{|\frac{k}{n}-x|<\frac{x}{2}} =: U_1 + U_2 + U_3.$$

Putting $M := \max_{[0,a]} |f|$ and using that $f \in F_1$, we have

$$(4) \qquad |f(x)| \leq |f(x)-f(a)| + |f(a)| \leq \frac{A}{a^{\beta/2}} (x-a)^{\beta} + M \leq K_1 x^{\beta} + M.$$

A similar estimate holds for $|f(-x)|$, therefore we get

$$|Z| \leq K_2 x^{\beta} e^{-2nx} \qquad (x \geq a).$$

In U_1 we have $|f(\frac{k}{n})| \leq M$. Hence, using (4) and (L4) we get

$$U_1 \leq (K_1 x^{\beta} + 2M) \sum_{|\frac{k}{n}-x|\geq\frac{x}{2}} p_{n,k}(x) \leq (K_1 x^{\beta} + 2M) \frac{4}{nx} \leq \left(\frac{K_1}{a^{1-\beta}} + 2\frac{M}{a} \right) \frac{4}{n}.$$

To estimate U_2 we consider $\frac{k}{n} \geq \frac{3x}{2}$, $f \in F_1$ and use (L3):

$$U_2 \leq \frac{A}{x^{\beta/2}} \sum_{|\frac{k}{n}-x|\geq\frac{x}{2}} \left|\frac{k}{n} - x\right|^{\beta} p_{n,k}(x) \leq \frac{A}{x^{\beta/2}} \left(\frac{x}{n}\right)^{\beta/2} = An^{-\beta/2}.$$

In U_3 we have $|f(\frac{k}{n}) - f(x)| \leq A\left|\frac{k}{n}-x\right|^{\beta} \Big/ \left(\frac{x}{2}\right)^{\beta/2}$. Hence, using (L3) we have

$$U_3 \leq A\left(\frac{2}{x}\right)^{\beta/2} \sum_{|\frac{k}{n}-x|<\frac{x}{2}} \left|\frac{k}{n} - x\right|^{\beta} p_{n,k}(x) \leq A\left(\frac{2}{x}\right)^{\beta/2} \left(\frac{x}{n}\right)^{\beta/2} = A\left(\frac{2}{n}\right)^{\beta/2}.$$

Then we have $|U| \leq K_2 n^{-\beta/2}$ with a K_2 independent of x $(x \geq a)$. The sum V can be estimated in the same way.

8. The proof of Theorem (T_2). If $g \in F_2$, then

$$|g(y)-g(t)| \leq |g'(\xi)|(y-t) \leq \frac{A}{\sqrt{\xi}} (y-t) \leq A \frac{y-t}{\sqrt{t}} \qquad (\tfrac{a}{2} < t < y < \infty),$$

therefore $g \in F_1$ with $\beta = 1$. Hence Theorem (T_1) implies (T_2).

9. The proof of Theorem (T_3). Let $x \geq a$, where $a \geq 0$. Since $H_n(f;0) = f(0)$, we can suppose that $x > 0$.

$$|U| \leq U_1 + U_2 + U_3^*, \text{ where } U_3^* := \left| \sum_{|\frac{k}{n}-x| < \frac{x}{2}} [f(\tfrac{k}{n}) - f(x)] p_{n,k}(x) \right|$$

$(U, U_1, U_2$ are defined in section 6 and 7.) The estimation of Z and U_1 is considered for $a > 0$ and $a = 0$ separately.

For $a > 0$, from $f \in F_3$ we obtain

$$(5) \quad |f(x)| \leq |f(x)-f(a)| + |f(a)| \leq |f'(\xi)|(x-a) + M \leq Bx + M.$$

A similar estimate holds for $|f(-x)|$, therefore we get

$$|Z| \leq K_3 x e^{-2nx} .$$

From (5), (L4) and $a > 0$ as in section 7 we obtain

$$U_1 \leq (Bx + 2M) \frac{4}{nx} \leq (4B + \frac{8M}{a}) \frac{1}{n} \leq K_4 \frac{1}{n} .$$

For $a = 0$ we have

$$|Z| \leq \{ |f(-x)-f(0)| + |f(0)-f(x)| \} e^{-2nx} \leq$$

$$\leq \{ |f'(\xi_1)| + |f'(\xi_2)| \} x e^{-2nx} \leq Bxe^{-2nx} \leq K_5 \frac{1}{n} .$$

For $a = 0$ the sum U_1 can be estimated in the same way as the sum U_2 below.

In U_2 we have $\left| f(\frac{k}{n}) - f(x) \right| \leq \left| f'(\xi_{n,k})(\frac{k}{n}-x) \right| \leq B\left|\frac{k}{n}-x\right|$.

Hence, and from (L5) we get $U_2 \leq 2B/n$.

$$U_3^* \leq \left| \sum_{|\frac{k}{n}-x| < \frac{x}{2}} f'(x)(\tfrac{k}{n}-x) p_{n,k}(x) \right| + \left| \sum_{|\frac{k}{n}-x| < \frac{x}{2}} \left[f'(\xi_{n,k}) - f'(x) \right] (\tfrac{k}{n}-x) p_{n,k}(x) \right| =:$$

$$=: U_{31}^* + U_{32}^* , \text{ where } x < \xi_{n,k} < \frac{k}{n} \text{ or } \frac{k}{n} < \xi_{n,k} < x.$$

By (L6) we have $U_{31}^* = |f'(x)| \left| \sum_{|\frac{k}{n}-x| < \frac{x}{2}} (\frac{k}{n}-x) p_{n,k}(x) \right| \leq 2B/n$.

From the first inequality of (C_3) it follows that

$$U_{32}^* \leq A\left(\frac{2}{x}\right)^{(1+\delta)/2} \sum_{|\frac{k}{n}-x| < \frac{x}{2}} |\xi_{n,k}-x|^{\delta} |\tfrac{k}{n}-x| p_{n,k}(x) .$$

From the inequality $|\xi_{n,k}-x| \leq |\frac{k}{n} - x|$ and (L3) we get

$$U^*_{32} \leq A\left(\frac{2}{x}\right)^{(1+\delta)/2} \left(\frac{x}{n}\right)^{(1+\delta)/2} = K_6 n^{-(1+\delta)/2} .$$

The sum V can be estimated in the same way as the sum U, however in the case of V^*_{31} we use (L7) instead of (L6).

10. The proof of Theorem (T_4). If $g \in F_4$, then

$$|g'(y)-g'(t)| = |g''(\xi)(y-t)| \leq \frac{A}{\xi}(y-t) \leq A\,\frac{y-t}{t} \qquad \left(\frac{a}{2} < t < y < \infty\right),$$

therefore $g \in F_3$ with $\delta = 1$. Hence Theorem (T_3) implies (T_4).

11. Remark. The proof of the Theorems shows that

$$f \in F_j \;\Rightarrow\; \sup_{x \geq a} |S_n(f;x) - f(x)| = O\left(n^{-\alpha_j}\right) \qquad (n \to \infty),$$

where $j = 1,2,3,4$.

References

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