КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 376—381

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p. 376-381

OPTIMAL QUADRATURE FORMULAE FOR DIFFERENTIABLE FUNCTIONS

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1. Introduction. Let

(1) $$Ly := y^{(r)} + a_{r-1}(x)y^{(r-1)} + \dots + a_0(x)y, \qquad a \leq x \leq b,$$

be a given linear differential operator with coefficients $a_j \in C^{(j)}[a, b]$, $j = 0, \dots, r-1$. Define the classes

$$LW_q^r[a, b] := \{ f \in C^{(r-1)}[a, b] \; : f^{(r-1)} \text{ abs. cont.}, \|Lf\|_q \leq 1 \}.$$

Given the multiplicities $(\nu_k)_1^n$, $1 \leq \nu_k \leq r$, and some specified conditions $\mathcal{B}$ on the boundary terms, we study here the question of existence and characterization of a quadrature formula (q.f.) of the form

(2) $$I(f) := \int_a^b f(x)dx \approx \sum_{j=0}^{r-1} A_j f^{(j)}(a) + \sum_{j=0}^{r-1} B_j f^{(j)}(b) +$$
$$+ \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} a_{k\lambda} f^{(\lambda)}(x_k) =: Q(\mathcal{B}; f)$$

with minimal error in $LW_q^r[a, b]$ for $1 < q \leq \infty$.

Minimizing the error

$$R_q(\bar{a}, \bar{x}) := \sup \{ |I(f) - Q(\mathcal{B}; f)| \; : f \in LW_q^r[a, b] \}$$

of (2) we treat as free parameters not only the coefficients A_j, B_j, $a_{k\lambda}$ but also the location of the nodes $a < x_1 < \dots < x_n < b$. The extremal q.f. is called optimal of type $(\nu_1, \dots, \nu_n; \mathcal{B})$.

The following two important kinds of boundary conditions $\mathcal{B}$ are covered:

(i) Periodic Boundary Conditions (PBC)

$$A_j = -B_j, \quad j = 0, \dots, r-1.$$

(ii) Zero Boundary Conditions (ZBC)

$$A_j = 0 \ , \quad j = \nu, \nu+1, \ \dots \ , r-1 \quad (0 \le \nu \le r)$$

$$B_j = 0 \ , \quad j = \mu, \mu+1, \ \dots \ , r-1 \quad (0 \le \mu \le r).$$

Denote by $\Omega(\nu_1, \dots, \nu_n)$ the set of all systems

$$(3) \qquad \bar{x} = \begin{pmatrix} x_1, \dots, x_n \\ \nu_1, \dots, \nu_n \end{pmatrix}$$

of points $(x_k)_1^n$, $a < x_1 < \dots < x_n < b$, of multiplicities $\nu_1, \dots, \nu_n$, respectively. The nodes $\bar{x}$ are said to be optimal of type $(\nu_1, \dots, \nu_n; \mathcal{B})$ in $LW_q^r[a, b]$ if

$$R_q(\bar{x}) = \inf \left\{ R_q(\bar{z}) : \bar{z} \in \Omega(\nu_1, \dots, \nu_n) \right\},$$

where

$$R_q(\bar{z}) := \inf \left\{ R_q(\bar{a}, \bar{z}) : \text{over all systems } \bar{a} \text{ of coefficients } \{a_{k\lambda}\} \text{ and } \{A_j\}, \{B_j\}, \text{ satisfying the conditions } \mathcal{B} \right\}.$$

We assume in this paper that the operator L has the property W of Polya (see [7] ,p.426) in $[a, b]$. Then the null space N_L of L is an Extended Complete Tchebycheff (ECT) space and there exist r strictly positive functions $w_i \in C^{(r)}[a, b]$, $i = 1, \dots r$, such that L can be factorized as

$$Lf = D_r D_{r-1} \dots D_1 f \ ,$$

where the differential operators D_i are defined by

$$D_i(f(x)) := w_i(x) \frac{d}{dx} (f(x)/w_i(x)) \ , \quad i = 1, \dots, r$$

(see [4],p.379). Denote, for convenience $D^{[j]} := D_j D_{j-1} \dots D_1$, $j = 1, \dots ,r$.

Following the method developed by B.Bojanov in [1,2,3] , we prove the existence of optimal quadrature formulae of arbitrary fixed type $(\nu_1, \dots, \nu_n; \mathcal{B})$ and give complete characterization of their coefficients. Our results extend those of K.Oskolkov [6] and M.Chahkiev [8], who treat the special case of L with characteristic polynomial having only real zeros, $\nu_1 = \dots = \nu_n = 1$, $\mathcal{B}$ = PBC. In the next sections we sketch the main steps of the proof.

2. Quadrature formulae and L-monosplines.

The proof of our existence theorem relies upon the connection between L-monosplines of least Lp- deviation and optimal quadrature formulae in the classes $LW_q^r[a, b]$. We assume that $1/q + 1/p = 1$, $1 \le p < \infty$.

We briefly describe this connection below. Consider the L-monospline $S(t)$ of the form

$$(4)\quad S(t) = \int_t^x G_L(\tau;t)d\tau - \sum_{j=0}^{r-1} B_j G_L^{(j)}(b;t) - \sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1} a_{k\lambda} G_L^{(\lambda)}(x_k;t),$$

where $G_L(x;y)$ is the Green's function associated with L (see [7], p.424) and boundary conditions $\mathcal{B}$ of the type

$$(5)\quad \begin{array}{ll} \text{(i)} & S^{(j)}(a) = S^{(j)}(b)\ ,\ j = 0, \dots , r-1 \quad \text{(PBC)}\ , \\ \text{(ii)} & S^{(j)}(a) = 0\ ,\ j = 0, \dots , r-\nu-1, \\ & S^{(j)}(b) = 0\ ,\ j = 0, \dots , r-\mu-1 \quad \text{(ZBC)}. \end{array}$$

Suppose that $f \in LW_q^r[a, b]$. Integration by parts produces the identity

$$(6)\quad \int_a^b S(t)Lf(t)dt = \int_a^b f(t)dt - \sum_{j=0}^{r-1} A_j f^{(j)}(a) - \sum_{j=0}^{r-1} B_j f^{(j)}(b) - \sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1} a_{k\lambda} f^{(\lambda)}(x_k)\ .$$

This expression suggest a quadrature formula of the type (2). Let $\mathcal{S}(\bar{x};\mathcal{B})$ denote the collection of L-monosplines $S(t)$ with knots $\bar{x}$, satisfying the boundary conditions $\mathcal{B}$. Now suppose that $S \in \mathcal{S}(\bar{x};\mathcal{B})$. Then (6) induces a q.f. of the type (2) which is exact for all $f \in N_L$. Conversely, let (2) be given with arbitrary fixed coefficients A_j, B_j, $a_{k\lambda}$, satisfying the relation $I(f) = Q(\mathcal{B};f)$ for $f \in N_L$. Then there exists $\tilde{S} \in \mathcal{S}(\bar{x};\mathcal{B})$ in the form (4)-(5) and the remainder term of formula (2) is $\int_a^b S(t)Lf(t)dt$. Formula (2) can be expressed in terms of $D^{[i]}$ derivatives, since by Leibniz's rule, there exist $c_{io}, \dots, c_{ii-1}$ such that for all f

$$D^{[i]}f(x) = f^{(i)}(x) + \sum_{j=0}^{i-1} c_{ij}(x) f^{(j)}(x)\ .$$

Then the resulting q.f. will be of the form

$$(7)\quad Q(\mathcal{B};f) = \sum_{j=0}^{r-1} \alpha_j D^{[j]}f(a) + \sum_{j=0}^{r-1} \beta_j D^{[j]}f(b) + \sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1} \ell_{k\lambda} D^{[\lambda]}f(x_k)\ .$$

Using Peano's remainder formula (see [5], p.10), the L-monospline $S(t)$ corresponding to this q.f. can be written as

$$(8)\quad S(t) = \int_t^b G_L(\tau;t)d\tau - \sum_{j=0}^{r-1}\beta_j D^{[j]}G_L(b;t) - \sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}\ell_{k\lambda}\, D^{[\lambda]}G\,(x_k;t)\,.$$

It is clear from the described one-to-one correspondence between q.f. and L-monosplines that

$$R_q(\bar{x}) = \inf\Big\{\sup\Big\{\Big|\int_a^b S(t)Lf(t)dt\Big| : f \in LW_q^r[a,b]\Big\} : S \in \mathcal{S}(\bar{x};\mathcal{B})\Big\} = \min\Big\{\|S\|_p : S \in \mathcal{S}(\bar{x};\mathcal{B})\Big\} = \|S(\bar{x};\cdot)\|_p\,.$$

Moreover, the existence of an optimal quadrature formula of the type $(\nu_1, \ldots, \nu_n; \mathcal{B})$ in $LW_q^r[a,b]$ is equivalent to the existen- of a L-monospline of least L_p deviation of the type $(\nu_1, \ldots, \nu_n; \mathcal{B})$.

3. Characterization results and existence. Denote for convenience,

$$LW_o(\bar{x};\mathcal{B}) := \Big\{f \in LW_q^r[a,b] : f \text{ satisfies PBC or ZBC},\ f^{(\lambda)}(x_k) = 0,\ k = 1, \ldots, n,\ \lambda = 0, \ldots, \nu_k-1\Big\}.$$

The following lemmas characterize the optimal quadrature formula.

Lemma 1. Given $\mathcal{B}$ and $1 < q \leq \infty$, for every fixed system of nodes $\bar{x} \in \Omega(\nu_1, \ldots, \nu_n)$ there exists a unique function $F(\bar{x};t) \in LW_o(\bar{x};\mathcal{B})$, such that $I(F(\bar{x};\cdot)) = R_q(\bar{x})$.

Lemma 2. Let $1 \leq p < \infty$. Suppose that the knots (3) are optimal of the type $(\nu_1, \ldots, \nu_n; \mathcal{B})$. Then

$$a_{k,\nu_k-1}F^{(\nu_k)}(\bar{x};x_k) = 0, \quad \text{for } \nu_k < r \text{ and}$$
$$S(\bar{x};x_k-0) = S(\bar{x};x_k+0), \quad \text{for } \nu_k = r, \text{ if } r \text{ is even}.$$

Denote the number of zeros of S in (a,b), counting multiplicities, by $Z(S;(a,b))$. Let us also set $\sigma_k = 1$ if ν_k is odd, and zero otherwise.

Lemma 3. Let $S(t)$ be an arbitrary L-monospline of the form (8) with $a < x_1 < \ldots < x_n < b$. Then

$$Z(S;(a,b)) \leq r + \sum_{k=1}^{n}(\nu_k + \sigma_k) - S^+(S(b), D^{[1]}S(b), \ldots, D^{[r]}S(b)) - S^+(S(a), -D^{[1]}S(a), \ldots, (-1)^r D^{[r]}S(a)).$$

Further,

$$(9)\quad Z(S;(-\infty,+\infty)) \leq r + \sum_{k=1}^{n}(\nu_k + \sigma_k).$$

Moreover, when equality holds in (9), the next two statements are valid :

If ν_k is odd then $\ell_{k\lambda} > 0$, $\lambda = 0, 2, \dots, \nu_k - 1$,

$(-1)^r D^{[i]} S(t) > 0$, $i = 0, \dots, r$, for $t >$ the last zero,

$(-1)^i D^{[i]} S(t) > 0$, $i = 0, \dots, r$, for $t <$ the first zero.

The proof of this result can be seen in [7], p.404.

Lemma 4. Let the multiplicities $(\nu_k)_1^n$ satisfy the inequalities $1 \leq \nu_k \leq r$, $k = 1, \dots, n$ and the following condition

(10) ν_k is even if $\nu_k < r$.

Then, for every system of knots $\bar{x} \in \Omega(\nu_1, \dots, \nu_n)$ the L-monospline $S(\bar{x};t)$ has a maximal number of zeros.

Lemma 5. Let the multiplicities ν_k satisfy the condition (10). Then

$F(\bar{x};t) \geq 0$, $t \in [a, b]$; $F(\bar{x};t) > 0$, $t \notin (a, x_1, \dots, x_n, b)$,

$F^{(\nu_k)}(\bar{x};x_k) > 0$ for $\nu_k < r$,

$F^{(\nu)}(\bar{x};a) > 0$, $(-1)^{\mu} F^{(\mu)}(\bar{x};b) > 0$, if $\mathcal{B} = ZBC$.

Now, using these lemmas, it is seen as in [1], that the following theorems hold.

Theorem 1. Let $1 < q \leq \infty$ and let $(\nu_k)_1^n$ be arbitrary fixed integer numbers satisfying the inequalities $1 \leq \nu_k \leq r$, $\nu + \mu + \nu_1 + \dots + \nu_n \geq r$. Then there exists an optimal quadrature formula of the form (2) (and (7)) in the class $LW_q^r[a, b]$. Moreover, the nodes and the coefficients of this q.f. satisfy the relations

$$
(11) \quad \begin{cases} a < x_1 < \dots < x_n < b, \\ \left.\begin{aligned} &\ell_{k,\nu_k-1} = 0, \ \ell_{k\lambda} > 0, \ \lambda = 0, 2, \dots, \nu_k - 2, \\ &a_{k,\nu_k-1} = 0, \ a_{k,\nu_k-2} > 0, \end{aligned}\right\} \text{if } \nu_k \text{ is even}, \\ \ell_{k\lambda} > 0, \ \lambda = 0, 2, \dots, \nu_k - 1, \ a_{k,\nu_k-1} > 0, \text{ if } \nu_k \text{ is odd}, \end{cases}
$$

$$
\alpha_j > 0, \ j = 0, \dots, \nu - 1, \ A_{\nu-1} > 0,
$$

$$
(-1)^j \beta_j > 0, \ j = 0, \dots, \mu - 1, \ (-1)^{\mu-1} B_{\mu-1} > 0.
$$

Theorem 2. Let $1 < q \leq \infty$ and let $(\nu_k)_1^n$ be arbitrary fixed integer numbers satisfying the inequalities $1 \leq \nu_k \leq r$, $\nu_1 + \dots + \nu_n \geq 1$. Then there exists an optimal quadrature formula of the form

$$
I(f) \approx \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} \ell_{k\lambda} D^{[\lambda]} f(x_k) \quad \left(\text{or } I(f) \approx \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} a_{k\lambda} f^{(\lambda)}(x_k)\right).
$$

in the class $L\tilde{W}_q^r[a, b] := \{ f \in LW_q^r[a, b] : f \text{ is } (b-a) \text{ - periodic} \}$.

Moreover, the coefficients of the optimal quadrature formula satisfy (11).

Using the relation between quadrature formulae and L-monosplines one can be reformulate Theorems 1 and 2 as existence theorems for L-monosplines of a preassigned type with minimal L_p- norm.

Acknowledgement. I wish to thankfully acknowledge the constant attention and valuable suggestions of Dr. B. Bojanov in the course of preparation of this paper.

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