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CAUCHY PROBLEM AND MODIFIED LACUNARY SPLINE FUNCTIONS

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In this paper an approximate solution is given for the Cauchy problem

(1) $$y''(x) = f[x,y(x),y'(x)] \qquad x \in [0,1]$$
$$y(0) := y_o, \quad y'(0) := y'_o;$$

with the help of lacunary spline functions of type (0,2,3), supposing

(2) $$f(x,y,y') \in C^3([0,1]xRxR), \text{ that is } y(x) \in C^5([0,1])$$

(3) $$|f^{(q)}(x,y_1,y'_1)-f^{(q)}(x,y_2,y'_2)| \leq L\{|y_2-y_1|+|y'_2-y'_1|\},$$
$$(q=0,1,2,3).$$

The main point of the method is that approximated values of $y^{(q)}(x_k)$ are constructed with the help of y_o, y'_o and function f, then using these approximate values $\bar{y}_k^{(q)}$ $(k=0,1,\dots,n;\ q=0,1,\dots,5)$ the solution of (1) and its derivatives are approximated up to the fifth order /except on the last subinterval where it is up to the third order/ by modified lacunary spline function of the type (0,2,3).

<u>Definition of the approximate values $\bar{y}_k^{(q)}$</u>:

Let $x_k = \frac{k}{m}$; $h = \frac{1}{m}$, $\omega(h,y^{(5)}) = \max_{|x-x_1|\leq h} |y^{(5)}(x)-y^{(5)}(x_1)|$;

$$\bar{y}_o := y_o, \quad \bar{y}'_o := y'_o, \quad \bar{y}_o^{(2+q)} := f^{(q)}(x_o,y_o,y'_o), \quad (q=0,1,2,3),$$

$$\bar{y}_k^{(q)} := G_k^{(q)}(x_k), \quad (q=0,1,\dots,5;\ k=1,2,\dots,m-1),$$

$$\bar{y}_m^{(q)} := G_{m-1}^{(q)}(x_m), \quad (q=0,1,\dots,5),$$

where $G_o(x) := y_o+y_o'(x-x_o)+\sum_{j=o}^{3} \frac{f^{(j)}[x_o,y_o,y_o']}{(j+2)!}(x-x_o)^{j+2}$,

$$G_k(x) := G_{k-1}(x_k)+G_{k-1}'(x_k)(x-x_k) + \\ + \sum_{j=o}^{3} \frac{f^{(j)}[x_k,G_{k-1}(x_k),G_{k-1}'(x_k)]}{(j+2)!}(x-x_k)^{j+2}.$$

It is proved in [1] /theorems 2.1 - 2.3/ by the author that

$$|y_k^{(q)}-\overline{y}_k^{(q)}| \leq C_q h^3\cdot\omega(h,y^{(5)}), \quad (q=0,1,\ldots,5;\ k=0,1,\ldots,m),$$

where constant C_q is independent of m.

Using these approximate values $\overline{y}_k^{(q)}$ $(q=0,2,3;\ k=0,1,\ldots,m)$ and $\overline{y}_o'$, $\overline{y}_m'$, on the bases of [2] we construct the lacunary spline function $\overline{S}_\Delta(x)$ of the type $(0,2,3)$ $(\overline{S}_\Delta(x) := \overline{S}_k(x)$ if $x_k \leq x \leq x_{k+1})$ and prove the following theorems.

<u>Theorem 1</u>. Let $\overline{y}_k^{(q)}$ $(q=0,1,2,3;\ k=0,1,\ldots,m)$ be the approximate values defined above. Then there exists a unique spline function $\overline{S}_\Delta(x)$ satisfying:

a./ $\overline{S}_o^{(j)}(x_o) := \overline{y}_o^{(j)}$, $(j=0,1,2,3)$,

b./ $\overline{S}_k^{(q)}(x_k) := \overline{y}_k^{(q)}$, $(q=0,2,3;\ k=1,2,\ldots,m-2)$,

c./ $\overline{S}_{m-1}^{(j)}(x_m) := \overline{y}_m^{(j)}$, $(j=0,1,2,3)$,

d./ $\overline{S}_k^{(q)}(x_{k+1}) := \overline{S}_{k+1}^{(q)}(x_{k+1}) := \overline{y}_{k+1}^{(q)}$, $(q=0,2,3;\ k=0,1,\ldots,m-2)$,

e./ $\overline{S}_\Delta(x)$ is a polynomial of minimal degree on $[x_k,x_{k+1}]$.

<u>Theorem 2</u>. If $y(x) \in C^5([0,1])$ is the exact solution of (1), and $\overline{S}_\Delta(x)$ is the spline function mentioned in Theorem 1, then the following inequalities hold:

$$|y^{(q)}(x)-\bar{S}_o^{(q)}(x)| \leq M_{o,q}h^{5-q}\omega(h,y^{(5)}), \quad (q=0,1,\ldots,5), \quad \text{if } x_o \leq x \leq x_1,$$

$$|y^{(q)}(x)-\bar{S}_k^{(q)}(x)| \leq M_{k,q}h^{3-q}\omega(h,y^{(5)}), \quad (q=0,1;\ k=1,2,\ldots,m-2), \quad \text{if } x_k \leq x \leq x_{k+1},$$

$$|y^{(q)}(x)-\bar{S}_k^{(q)}(x)| \leq M_{k,q}h^{5-q}\omega(h,y^{(5)}), \quad (q=2,3,4,5;\ k=1,2,\ldots,m-2), \quad \text{if } x_k \leq x \leq x_{k+1},$$

$$|y^{(q)}(x)-\bar{S}_{m-1}^{(q)}(x)| \leq M_{m-1,q}h^{3-q}\omega(h,y^{(5)}), \quad (q=0,1,\ldots,5), \quad \text{if } x_{m-1} \leq x \leq x_m,$$

where the constants $M_{k,q}$ are independent of m.

Theorem 3. If the function f in (1) statisfies conditions (2) and (3), then the following inequalities hold:

$$|\bar{S}_o''(x)-f[x,\bar{S}_o(x),\bar{S}_o'(x)]| \leq M^*_{o,2}h^3\omega(h,y^{(5)}), \quad \text{if } x_o \leq x \leq x_1,$$

$$|\bar{S}_k''(x)-f[x,\bar{S}_k(x),\bar{S}_k'(x)]| \leq M^*_{k,2}h^2\omega(h,y^{(5)}), \quad \text{if } x_k \leq x \leq x_{k+1}, \quad (k=1,2,\ldots,m-2),$$

$$|\bar{S}_{m-1}''(x)-f[x,\bar{S}_{m-1}(x),\bar{S}_{m-1}'(x)]| \leq M^*_{m-1,2}h\omega(h,y^{(5)}), \quad \text{if } x_{m-1} \leq x \leq x_m,$$

where the constants $M^*_{k,2}$ are independent of m.

Proof of Theorem 1. Theorem 2.1 of [2] implies Theorem 1, and on the bases of it $\bar{S}_\Delta(x)$ is the following:

$$\bar{S}_o(x) = \bar{y}_o+\bar{y}_o'(x-x_o)+\frac{\bar{y}_o''}{2}(x-x_o)^2+\frac{\bar{y}_o'''}{6}(x-x_o)^3 + \\ + \bar{a}_{o,4}(x-x_o)^4+\bar{a}_{o,5}(x-x_o)^5+\bar{a}_{o,6}(x-x_o)^6,$$

$$\bar{S}_k(x) = \bar{y}_k+\bar{a}_{k,1}(x-x_k)+\frac{\bar{y}_k''}{2}(x-x_k)^2+\frac{\bar{y}_k'''}{6}(x-x_k)^3+\bar{a}_{k,4}(x-x_k)^4 + \\ + \bar{a}_{k,5}(x-x_k)^5,$$

$$\bar{S}_{m-1}(x) = \bar{y}_{m-1}+\bar{a}_{m-1,1}(x-x_k)+\frac{\bar{y}''_{m-1}}{2}(x-x_{m-1})^2+\frac{\bar{y}'''_{m-1}}{6}(x-x_{m-1})^3+$$

$$+\bar{a}_{m-1,4}(x-x_{m-1})^4+\bar{a}_{m-1,5}(x-x_{m-1})^5+\bar{a}_{m-1,6}(x-x_{m-1})^6.$$

The coefficients of the polynomials are also given in [2].

Using the formulas of the coefficients, and the theorems 2.1-2.3 of [1], the following lemmas, needed for the proof of Theorem 2, can be easily proved.

Lemma 1. Let $a_{k,j}$ and $\bar{a}_{k,j}$ denote the coefficients of the lacunary spline function $S_\Delta(x)$ of the type (0,2,3) constructed by the "true" values involved in [2] and the ones in the proof of Theorem 1 respectively. Then we have

$$|a_{0,j}-\bar{a}_{0,j}| \leq B_{0,j}h^{5-j}\omega(h,y^{(5)}), \quad (j=4,5,6),$$

$$|a_{k,1}-\bar{a}_{k,1}| \leq B_{k,1}h^{2}\omega(h,y^{(5)}), \quad (k=1,2,\ldots,m-2),$$

$$|a_{k,j}-\bar{a}_{k,j}| \leq B_{k,j}h^{5-j}\omega(h,y^{(5)}), \quad (j=3,4;\ k=1,2,\ldots,m-2),$$

$$|a_{m-1,j}-\bar{a}_{m-1,j}| \leq B_{m-1,j}h^{3-j}\omega(h,y^{(5)}), \quad (j=1,4,5,6),$$

where the constants $B_{k,j}$ are independent of m.

Lemma 2. Let $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ denote the lacunary spline function of the type (0,2,3) constructed by the "true" values /Theorem 2.1. of [2]/, and the one constructed by the approximate values /Theorem 1/ respectively. Then we have

$$|S_0^{(q)}(x)-\bar{S}_0^{(q)}(x)| \leq K_{0,q}h^{5-q}\omega(h,y^{(5)}), \quad (q=0,1,\ldots,6), \quad \text{if } x_0 \leq x \leq x_1,$$

$$|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)| \leq K_{k,q}h^{3-q}\omega(h,y^{(5)}), \quad (q=0,1;\ k=1,2,\ldots,m-2), \quad \text{if } x_k \leq x \leq x_{k+1},$$

$$|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)| \leq K_{k,q}h^{5-q}\omega(h,y^{(5)}), \quad (q=2,3,4,5;\ k=1,2,\ldots,m-2), \quad \text{if } x_k \leq x \leq x_{k+1},$$

$$|S^{(q)}_{m-1}(x)-\bar{S}^{(q)}_{m-1}(x)| \leq K_{m-1,q}h^{3-q}\omega(h,y^{(5)}), \quad (q=0,1,\ldots,6), \quad \text{if} \quad x_{m-1} \leq x \leq x_m ,$$

where the constants $K_{k,q}$ are independent of m.

Prof of Theorem 2. Lemma 2, Theorem 3.1. of [2] and the inequality

$$|y^{(q)}(x)-\bar{S}^{(q)}_k(x)| \leq |y^{(q)}(x)-S^{(q)}_k(x)| + |S^{(q)}_k(x)-\bar{S}^{(q)}_k(x)|$$

imply Theorem 2.

Prof of Theorem 3. Using conditions (1), (2) and (3) we obtain

$$|\bar{S}''_\Delta(x)-f[x,\bar{S}_\Delta(x),\bar{S}'_\Delta(x)]| \leq |\bar{S}''_\Delta(x)-y''(x)| + |y''(x)-f[x,\bar{S}_\Delta(x),\bar{S}'_\Delta(x)]| \leq$$

$$\leq |\bar{S}''_\Delta(x)-y''(x)| + |f[x,y(x),y'(x)] - f[x,\bar{S}_\Delta(x),\bar{S}'_\Delta(x)]| \leq$$

$$\leq |\bar{S}''_\Delta(x)-y''(x)| + L\{|y(x)-\bar{S}_\Delta(x)| + |y'(x)-\bar{S}'_\Delta(x)|\} ,$$

that implies Theorem 3 with the help of Theorem 2.

References

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