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CLOSED PRIMARY IDEALS AND POINT DERIVATIONS IN ZYGMUND ALGEBRAS

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1. Introduction. Let Q be the standard closed unit cube in R^n. We write $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^2 f(x) = f(x-h) - 2f(x) + f(x+h)$, $|h| = \max_{1\le i\le n} |h_i|$, where f is a function on Q, $x\in Q$, $h=(h_1,\dots,h_n)\in R^n$ and $x\pm h\in Q$. Define $\omega_2(f;Q;t) = \sup\{|\Delta_h^2 f(x)| \mid x\in Q, |h|\le t\}$, $t\in R_+$, and $\|f\|_Q = \sup\{|f(x)| \mid x\in Q\}$. Denote $\Lambda=\Lambda(Q)$ the Zygmund space of all real functions f on Q with the finite norm $\|f\| = \|f\|_{\Lambda(Q)} = \max\{\|f\|_Q, \sup_{t>0}[\omega_2(f;Q;t)/t]\}$.

D.R.Sherbert [1] has described closed primary ideals and point derivations in the algebras $Lip(X,\rho)$ in the case (X,ρ) is a compact metric space. The descriptions depend largely on the fact that for all subsets F of X every real function $f\in Lip(F,\rho)$ may be extended in a norm preserving way to a function $\tilde f\in Lip(X,\rho)$.

In the present work analogous descriptions of closed primary ideals and of point derivations in the algebras $\Lambda(Q)$ are given. However the demonstrations prove to be more complicated in the case of Zygmund algebras than those for Lipschitz algebras. It is conditioned by the fact that, even if F is a closed cube contained in Q, the least constant C_F, such that the inequality $\inf\|\tilde f\|_{\Lambda(Q)} \le C_F \|f\|_{\Lambda(F)}$ (where inf is taken over all extensions $\tilde f\in\Lambda(Q)$ of a function f) hold up for all $f\in\Lambda(F)$, depends on F and, moreover, $C_F \longrightarrow \infty$ as $diam\, F\longrightarrow 0$. To overcome this difficulty, we approximate Zygmund functions by appropriate polynomials of degree not more than 1 and subject the latter to a special extension (see Section 4).

We show in Section 6 that point derivations play the same fundamental role in the theory of Zygmund algebras as in that of Lipschitz algebras. Every proposition found in this section has a counterpart in Lipschitz algebras.

All results presented below can be transferred to Zygmund algebras of complex functions.

2. Some preliminary observations. Let $f \in \Lambda$ and F be a subset of Q. We put $\|f\|_F = \sup\{|f(x)| \,|\, x \in F\}$ and denote $P(f;F)$ the polynomial of degree not more than 1 which has the least deviation from f in $L_\infty(F)$. The letter A denotes hereafter various positive constants that may depend only on n.

It is well known that if $S = S(a,d)$ is a closed cube centered at $a \in Q$ with the sides of length $2d \le 1$ parallel to the axes then

$$\|f - P(f; S\cap Q)\|_{S\cap Q} \le A\,\omega_2(f; S\cap Q; d). \tag{1}$$

By the Marchaud's inequality we have for all $x, x+h \in Q$ and $f \in \Lambda$

$$|f(x+h) - f(x)| \le A\|f\|\,|h| \ln\frac{2}{|h|}\,. \tag{2}$$

Hence $\Lambda \subset C(Q)$.

We point out the equality

$$\Delta_h^2(fg)(x) = \Delta_h^2 f(x)\, g(x+h) + 2\Delta_h^1 f(x-h)\,\Delta_h^1 g(x) + f(x-h)\,\Delta_h^2 g(x). \tag{3}$$

It follows from (3) and (2) that $\|fg\| \le A\|f\|\|g\|$ for all $f,g \in \Lambda$, i.e. Λ is a Banach algebra. It possesses the following inversion property: if $f \in \Lambda$ and $f(x) \ne 0$ for all $x \in Q$ then $1/f \in \Lambda$. Therefore its maximal ideal space is Q. Thus, Λ is a commutative semisimple regular Banach algebra with the unity $\mathbb{1}$.

3. First results on closed primary ideals. Let $x \in Q$. We denote $M(x)$ the maximal ideal of functions in Λ vanishing at x and $J(x)$ the minimal closed primary ideal at x which coinsides with the closure of the set of functions in Λ vanishing in a (relative) neighbourhood of x. If F is an arbitrary closed subset of Q, we define the maximal closed ideal $M(F)$ with the spectrum F and the minimal one $J(F)$ in a similar way.

Proposition 3.1. $J(x) = \operatorname{clos}_\Lambda[M(x)^2]$.

Proof. The inclusion $J(x) \subset \operatorname{clos}_\Lambda[M(x)^2]$ follows from the regularity of the algebra Λ. To establish the opposite inclusion, it is sufficient to show that $f \in M(x)$ imply $f^2 \in J(x)$. Let ω be a function in $C^\infty(R^n)$ with the properties $0 \le \omega \le 1$, $\omega \equiv 1$ on $S(x,\frac{1}{2})$ and $\omega \equiv 0$ outside $S(x,1)$. Set $\omega_\delta(x) = \omega(\frac{x}{\delta})$, $0 < \delta \le \frac{1}{2}$. We conclude from (3) and (2) that $\|f^2\omega_\delta\| \le A\|f\|^2 \delta \ln^2\frac{1}{\delta}$. It yields $\|f^2\omega_\delta\| \to 0$ as $\delta \to 0$. Since the functions $f^2 - f^2\omega_\delta$ vanish in a neighbourhood of x , $f^2 \in J(x)$.

It can be seen likewise that for every closed set F , $F \subset Q$, $J(F) = clos_\Lambda [M(F)^2]$.

As a corollary of Proposition 3.1 we obtain the following simple description of the closed primary ideals in Λ .

Proposition 3.2. A closed linear subspace $I \subset \Lambda$ is a closed primary ideal at a point $x \in Q$ if and only if $J(x) \subset I \subset M(x)$.

Proof. The "only if" part being trivial, we have to show that I is an ideal provided the above inclusions are fulfilled. Suppose $f \in I$ and $g \in \Lambda$, then $fg = f[g - g(x)] + g(x)f \in M(x)^2 + I \subset J(x) + I = I$.

4. The structure of ideals $J(x)$. We begin with the special extension lemma.

Lemma 4.1. Let $0<\delta<1$. There exist a function $\varphi_\delta \in \Lambda(R^1)$ such that $\varphi_\delta(x) = x$, if $|x| \le \delta$, and $\|\varphi_\delta\|_{\Lambda(R^1)} \le \dfrac{C}{\ln\frac{1}{\delta}}$, where C is a positive constant independent on δ .

Proof. Let r be the maximal integer with the property $2^{r-1}\delta \le 1$. Consider $\mu = \sigma_0 - \frac{1}{r}\sum_{i=0}^{r-1} \sigma_{2^i\delta}$, where σ_a is the Dirac measure at a point a . The formula $\varphi_\delta(x) = \int_0^x (x-t)\,d\mu(t)$, $x \in R_+$, defines a nondecreasing piecewise linear function such that $\varphi_\delta(x) = x$ when $0 \le x \le \delta$ and $\varphi_\delta(x) = \varphi_\delta(2^{r-1}\delta)$ when $x \ge 2^{r-1}\delta$. By extending it to R^1 in an odd manner we obtain the function desired.

Remark. There is a constant $\tilde{C} > 0$ independent on δ such that for every function $\tilde{\varphi}_\delta$ on R^1 with the property $\tilde{\varphi}_\delta(x) = x$, if $|x| \le \delta$, we have $\|\tilde{\varphi}_\delta\|_{\Lambda(R^1)} \ge \dfrac{\tilde{C}}{\ln\frac{1}{\delta}}$. Thus, Lemma 4.1 is sharp in a sense.

The ideals $J(x)$, $x \in Q$, turn out to have the following structure.

Theorem 4.2. $J(x) = \{ f \in \Lambda \mid f(x) = 0, \varlimsup_{\substack{y \to x \\ h \to 0}} [|\Delta_h^2 f(y)|/|h|] = 0 \}$. (4)

Proof. Denote $\mathcal{J}(x)$ the right-hand side of (4). The inclusion $J(x) \subset \mathcal{J}(x)$ is obvious. Conversely, let $f \in \mathcal{J}(x)$. Fix $\varepsilon > 0$. According to (4) there exist δ, $0<\delta\le\frac{1}{2}$, such that

$$|\Delta_h^2 f(y)| \le \varepsilon |h|, \text{ if } y, y\pm h\in Q, |y-x|\le\delta \text{ and } |h|\le\delta . \qquad (5)$$

Define $F=S(x,\delta)\cap Q$, $P=P(f;F)-P(f;F)(x)$ and $g=(f-P)\omega_\delta$, ω_δ being the same as in Section 3. (1) and (5) imply $\|f-P\|_F \le 2\|f-P(f;F)\|_F \le A\varepsilon\delta$. We deduce from the latter estimate and from (5) that $\|g\|\le A\varepsilon$. It follows from Lemma 4.1 that the functions $y\longmapsto y_i-x_i$, $y\in Q$, $i=1,\dots,n$, belong to $J(x)$. Since $P(x)=0$, P belongs to $J(x)$ as well. Consider $k=(f-f\omega_\delta)+P\omega_\delta$, $k\in J(x)$. From $f=g+k$ we see, via arbitrariness of ε, that $f\in J(x)$. The theorem is proved.

Denote $\pi:\Lambda\longrightarrow\Lambda/_{J(x)}$ the canonical homomorphism and put $\dot f=\pi f$. We indicate that the norm on $\Lambda/_{J(x)}$ provided by the functional $N(\dot f)=\max\{|f(x)|, \overline{\lim}_{\substack{y\to x\\ h\to 0}} [|\Delta_h^2 f(y)|/|h|]\}$ is equivalent to the usual norm $\|\dot f\|=\inf\{\|f+g\| \mid g\in J(x)\}$ on $\Lambda/_{J(x)}$, the proof being similar to that of Theorem 4.2.

5. Point derivations in Λ.

Definition 5.1. A functional $D\in\Lambda^*$ is called a point derivation at a point $x\in Q$ if

$$D(fg)=f(x)Dg+g(x)Df \quad \text{for all } f,g\in\Lambda .$$

Let $\mathcal{D}_x$ be the set of all point derivations at x and $\mathcal{D}=\bigcup_{x\in Q}\mathcal{D}_x$. Clearly $\mathcal{D}_x$ is a linear subspace of Λ^* closed in the weak * topology. For $y, y\pm h\in Q$, $h\ne 0$ we define the functional $\Psi_{y,h}(f)=\Delta_h^2 f(y)/|h|$, $f\in\Lambda$. Denote T_x the set of all weak * limits of the functionals $\Psi_{y,h}$ as $y\longrightarrow x$ and $h\longrightarrow 0$. It follows from (2) and (3) that $T_x\subset\mathcal{D}_x$.

Let $E\subset\Lambda$ and $G\subset\Lambda^*$. We denote by $E^\perp$ and ${}^\perp G$ their annihilators in Λ^* and in Λ respectively. The notation $V(G)$ will be used for the weak * closure of the linear span of G.

Theorem 5.2. $\mathcal{D}_x=V(T_x)$.

Proof. It is enough to show that $\mathcal{D}_x\subset V(T_x)$ or, if stated in a dual form, that ${}^\perp T_x\subset{}^\perp\mathcal{D}_x$. Suppose the latter inclusion fails.

Choose a function $f \in {}^{\perp}T_x \setminus {}^{\perp}\mathcal{D}_x$. We may assume without loss of generality that $f(x)=0$. We have $\mathcal{D}_x = [\mathbb{1} \cup M(x)^2]^{\perp}$, this formula being true in any commutative unital Banach algebra. Thus, in view of Proposition 3.1 ${}^{\perp}\mathcal{D}_x = \{t\mathbb{1}\}_{t\in R^1} + J(x)$; hence $f \notin J(x)$. By Theorem 4.2 there are $\varepsilon > 0$ and sequences $\{x_m\}, \{h_m\}$ such that $x_m, x_m \pm h_m \in Q$, $h_m \neq \mathbb{0}$, $|x_m - x| \leq \frac{1}{m}$, $|h_m| \leq \frac{1}{m}$ and $|\Delta^2_{h_m} f(x_m)|/|h_m| \geqslant \varepsilon$ for all $m \in \mathbb{N}$. Denote $\Psi_m = \Psi_{x_m, h_m}$. Since $\|\Psi_m\|_{\Lambda^*} \leq 1$, the set $\{\Psi_m\}_{m\in\mathbb{N}}$ has a limit point $\Psi \in T_x$. We note $|\Psi_m(f)| \geqslant \varepsilon$, therefore $|\Psi(f)| \geqslant \varepsilon$, i.e. $f \notin {}^{\perp}T_x$. The contradiction obtained means ${}^{\perp}T_x \subset {}^{\perp}\mathcal{D}_x$. The proof is completed.

It is worth noting that point derivations in Λ can also be described in terms of the Stone-Čech compactification K of the topological space $\mathcal{U} = \{(x,h) \mid x \in Q, h \in R^n \setminus \{\mathbb{0}\}, x \pm h \in Q\}$, see [2]. In fact, define for each $x \in Q$ the set K_x of all Moore-Smith limits in K of generalized sequences $\{(x_\alpha, h_\alpha)\} \subset \mathcal{U}$, such that $x_\alpha \longrightarrow x$ and $h_\alpha \longrightarrow \mathbb{0}$. We have $K \setminus \mathcal{U} = \bigcup_{x\in Q} K_x$, where all K_x are disjoint and nonempty. With a function $f \in \Lambda$ the (unique) extension $\hat{f} \in C(K)$ of the function $(x,h) \longmapsto \Delta^2_h f(x)/|h|$, $(x,h) \in \mathcal{U}$, is associated. For $z \in K_x$ we consider the functional $\theta_z(f) = \hat{f}(z)$, $f \in \Lambda$. Then θ_z is a point derivation at x, $T_x = \{\theta_z \mid z \in K_x\}$ and, by Theorem 5.2, $\mathcal{D}_x = V(\{\theta_z \mid z \in K_x\})$.

6. Applications of point derivations.

Let $\lambda = \lambda(Q)$ be the closed subspace of Λ consisting of functions f with the property $\omega_2(f;Q;t)/t \longrightarrow 0$ as $t \longrightarrow 0$.

Proposition 6.1. $\lambda = \{f \in \Lambda \mid Df = 0$ for all $D \in \mathcal{D}\}$.

Next we will characterize weak convergence of sequences in Λ.

Proposition 6.2. A sequence $\{f_m\}_{m\in\mathbb{N}} \subset \Lambda$ converges weakly to $f \in \Lambda$ if and only if the following conditions are satisfied:

(i) $\{f_m\}_{m\in\mathbb{N}}$ is bounded in Λ ;

(ii) $f_m(x) \longrightarrow f(x)$ as $m \longrightarrow \infty$ for all $x \in Q$;

(iii) $Df_m \longrightarrow Df$ as $m \longrightarrow \infty$ for all $D \in \mathcal{D}$.

Consider the set $\mathcal{L}_x = \{\ell\}$ of all linear subspaces in $\mathcal{D}_x$ closed in the weak * topology. Point derivations enable one more description of the set $\mathcal{J}_x$ of closed primary ideals at x (cf. Proposition 3.2).

Proposition 6.3. The mapping

$$\ell \longmapsto I_\ell = \{f \in \Lambda \mid f(x)=0,\ Df=0 \text{ for all } D \in \ell\}$$

provides one-to-one correspondence between $\mathcal{L}_x$ and $\mathcal{I}_x$. The inverse correspondence is given by the mapping

$$I \longmapsto \ell_I = \{D \in \mathcal{D}_x \mid Df=0 \text{ for all } f \in I\}.$$

The last application of point derivations concerns differentiations in Zygmund algebras.

Definition 6.4. A continious linear operator $\Delta : \Lambda \longrightarrow \Lambda$ is called a differentiation in Λ if

$$\Delta(fg) = \Delta f\, g + \Delta g\, f \quad \text{for all } f, g \in \Lambda.$$

Proposition 6.5. $\Delta = \mathbb{O}$ is the only differentiation in Λ .

The same result, though absent in [1] , is also valid for Lipschitz algebras.

References

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