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MIXED NORM MULTIVARIATE APPROXIMATION WITH BLENDING FUNCTIONS

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1. Approximation. Let $(X, \|.\|)$ be a normed vector space (of real functions), and $U \subset X$ a linear subspace. We consider the problem: Given $f \in X$, does there exist an $u^* \in U$ such that

$$(*) \quad \|f - u^*\| = E_U(f) := \inf_{u \in U} \|f - u\| \quad ?$$

Here $E_U(f)$ is called the approximation constant (degree) of f with respect to U. The subspace U is said to be proximinal, if for every $f \in X$ there exists a proximum (best approximation), i. e. an element $u^* = u^*(f)$ satisfying equation (*). A finite dimensional U is proximinal, and a proximinal U is closed. The infinite-dimensional case is particularly interesting in connection with multivariate functions, see the recent survey by Cheney [1].

Our main interest is concentrated upon approximation with respect to blending functions and mixed norms. We consider blending functions in the framework of tensor products. Let X and Y be two normed vector spaces (of functions), and let α be a cross-norm on $X \otimes Y$, and $X \otimes_\alpha Y$ the completion of $X \otimes Y$ with respect to α, see Gilbert-Leih [5]. Given subspaces U and V of X resp. Y, we define the space of blending functions:

$$B(U,X;V,Y) := U \otimes Y + X \otimes V.$$

This vector space has infinite dimension if $U \neq (0)$ and Y is infinite-dimensional (or if X is infinite-dimensional and $V \neq (0)$). In general it is not known whether $B(U,X;V,Y)$ is proximinal. Using maximal linear functionals we construct best approximants to functions of product type (Section 3). Special cases of our result are due to Ehlich-Zeller [3], Fromm [4], Reimer [9], and Steinhaus [11]. In Section 4 we shall characterize best approximants for general functions with respect to $\|.\|_{\infty,q}$, $1 < q < \infty$. The case $q = 1$ was dealt with by Cheney-McCabe-Phillips [2], see also Krabs [7].

2. Maximal linear functionals. We use the following characterization theorem (based on the Helly-Hahn-Banach principle):

Lemma 1. (Singer [10, p. 18])
Given the normed vector space X and the subspace $U \subset X$, $f \in X \setminus \bar{U}$, and $u^* \in U$. Then the following conditions are equivalent:

(i) u^* is a proximum for f with respect to U.

(ii) There exists a $\varphi \in X'$ (bounded linear functionals) such that
 (a) $\|\varphi\| = 1$,
 (b) $\varphi(u) = 0$ for all $u \in U$,
 (c) $\varphi(f) = \varphi(f-u^*) = \|f-u^*\|$.

φ is called a maximal linear functional for f.

We consider some examples of maximal linear functionals, where X consists of real functions g defined on a compact interval $I \subset \mathbb{R}$.

Chebyshev-norm $\|.\|_\infty$: Let $X = C(I)$, endowed with the Chebyshev-norm $\|g\|_\infty := \sup_{t \in I} |g(t)|$, U an n-dimensional Chebyshev subspace. Given $f \in C(I)$, then we have

$$\varphi(g) := \varphi_{f,\infty}(g) = \sum_{\mu=0}^{m} a_\mu g(x_\mu) \qquad \text{for } g \in C(I),$$

where the x_μ constitute a set of alternation points belonging to the (unique) proximum $u^* \in U$ to f, the a_μ are suitably chosen with $\Sigma|a_\mu| = 1$.

L_p-norms $\|.\|_p$: Here we have $X = L_p(I)$, $1 \le p < \infty$ (consisting of classes of functions) endowed with $\|g\|_p := (\int_I |g(t)|^p dt)^{1/p}$. For an arbitrary subspace U of $L_p(I)$ a maximal linear functional is given by

$$\varphi(g) := \varphi_{f,p}(g) = \int_I g(t)\beta(t)dt \qquad \text{for } g \in L_p(I)$$

with suitable weight function β, cf. Singer [10, p. 46 and pp. 56-57]. We distinguish two cases:

(1) $p = 1$: Here we have $\beta \in L_\infty(I)$, ess sup $|\beta| = 1$. Further, if $f-u^*$ vanishes only on a set of measure zero, then $\beta(t) = \text{sign}(f-u^*)(t)$.

(2) $1 < p < \infty$: Now one has $\beta \in L_{p'}(I)$, where $1/p' + 1/p = 1$, $\|\beta\|_{p'} = 1$. We have explicitely (for $u^* \neq f$)

$$\beta(t) = \text{sign}(f-u^*)(t) \cdot |f(t)-u^*(t)|^{p-1} / \|f-u^*\|_p^{p-1}.$$

For the following, let $I, J \subset \mathbb{R}$ be two compact intervals. Given an $h \in C(I \times J)$, then we introduce the partial function $h_x : J \to \mathbb{R}$ defined by $h_x(y) := h(x,y)$ for $x \in I$. The mapping $\omega : x \to h_x$ from I into $C(J)$ is continuous if we endow $C(J)$ with any L_q-norm $(1 \le q < \infty)$. We shall use this in Section 4.

3. Mixed norm approximation of product functions. Let $I, J \subset \mathbb{R}$ be compact intervals. Our main result is concerned with the construction of blending functions of best mixed norm approximation to functions of product type in $C(I\times J)$. Given subspaces U and V of $C(I)$ resp. $C(J)$, then the vector space of blending functions $B(U,C(I);V,C(J))$ is given by

$$\{w \mid w(x,y) = \Sigma\, u_i(x)g_i(y) + \Sigma\, f_j(x)v_j(y)\},$$

where $f_j \in C(I)$, $g_i \in C(J)$, $u_i \in U$, and $v_j \in V$ (see also [6]). We consider the following mixed norms: For $h \in C(I\times J)$ we define

$$\alpha_{\infty\infty}(h) := \|h\|_{\infty,\infty} := \sup_{x\in I} \sup_{y\in J} |h(x,y)|,$$

$$\alpha_{\infty q}(h) := \|h\|_{\infty,q} := \sup_{x\in I} \Big(\int_J |h(x,y)|^q dy\Big)^{1/q} \qquad (1 \le q < \infty),$$

$$\alpha_{pq}(h) := \|h\|_{p,q} := \Big(\int_J \Big(\int_I |h(x,y)|^p dx\Big)^{q/p} dy\Big)^{1/q} \qquad (1 \le p,q < \infty).$$

The meaning of $\alpha_{p\infty}$ $(1 \le p < \infty)$ is obvious. All norms α_{pq} $(1 \le p,q \le \infty)$ are cross-norms (see Gilbert-Leih [5]), and α_{pp} is the bivariate L_p-norm $(1 \le p < \infty)$.

Theorem 2.

Let $f \in C(I)$, $g \in C(J)$, and $H(x,y) = f(x)g(y)$. Suppose U and V are subspaces of $C(I)$ and $C(J)$, respectively, and u^* is a proximum to f with respect to U and $\|.\|_p$, and v^* is a proximum to g with respect to V and $\|.\|_q$ $(1 \le p,q \le \infty$; if p resp. q equals ∞, then U resp. V is assumed to be a finite dimensional Chebyshev subspace). Then we have:

$w^* := u^*g + fv^* - u^*v^*$ is a proximum to H with respect to the space of blending functions $B(U,C(I);V,C(J))$ in the α_{pq}-norm. Moreover for the approximation constant we have (superscript indicates norm)

$$E^{p,q}_{B(U,C(I);V,C(J))}(H) = E^p_U(f)\cdot E^q_V(g).$$

Proof. Let us consider the linear functionals $\lambda := \varphi_{f,p}$ and $\mu := \varphi_{g,q}$ according to Section 2 as well as $\sigma := \sigma_{H;p,q} := \lambda\otimes\mu$, which can be uniquely extended from $C(I)\otimes C(J)$ to $C(I\times J)$. This extension will be called σ, too. We are going to apply Lemma 1.

First we show by elementary calculation that condition (ii.a) holds true, i. e. $\|\sigma\| = 1$ in the operator norm induced by α_{pq} $(1 \le p,q \le \infty)$. Indeed, in the case of $\sigma_{\infty,\infty}$ we have

$$\sigma_{\infty,\infty}(h) = \sum_{\mu=o}^{m} \sum_{\nu=o}^{n} a_\mu b_\nu h(x_\mu,y_\nu), \qquad h \in C(I\times J),$$

where $\{x_o,\dots,x_m\}$ and $\{y_o,\dots,y_n\}$ are sets of alternation points associated with the one-dimensional approximation problems for f resp. g. This representation yields

$$\|\sigma_{\infty,\infty}\| \le \sum_{\mu=o}^{m}\sum_{\nu=o}^{n} |a_\mu|\cdot|b_\nu| = 1.$$

Next we consider $\sigma_{\infty,q}$ for $1 \le q < \infty$. The case $\sigma_{p,\infty}$, $1 \le p < \infty$, can be treated in the same way. We have

$$\sigma_{\infty,q}(h) = \sum_{\mu=o}^{m} a_\mu \int_J h(x_\mu,t)\beta(t)dt, \qquad h \in C(I\times J),$$

with β according to Section 2. Hence with the aid of Hölder's inequality (using the conjugate exponent q' given by $1/q' + 1/q = 1$),

$$\|\sigma_{\infty,q}\| \le \sup_{\|h\|_{\infty,q}\le 1} \sum_{\mu=o}^{m} |a_\mu|\cdot \int_J |h(x_\mu,t)|\cdot|\beta(t)|dt$$

$$\le \sup_{\|h\|_{\infty,q}\le 1} \sum_{\mu=o}^{m} |a_\mu|\cdot\left(\int_J |h(x_\mu,t)|^q dt\right)^{1/q}\cdot\|\beta\|_{q'}$$

$$\le \sum_{\mu=o}^{m} |a_\mu|\cdot\|\beta\|_{q'} = 1.$$

Finally, we consider the functional $\sigma_{p,q}$, $1 \le p,q < \infty$. We have

$$\sigma_{p,q}(h) = \int_I\int_J h(s,t)\beta_1(s)\beta_2(t)dsdt, \qquad h \in C(I\times J),$$

with β_1 and β_2 according to Section 2. Hölder's inequality yields

$$\|\sigma_{p,q}\| \le \sup_{\|h\|_{p,q}\le 1} \int_I\int_J |h(s,t)|\cdot|\beta_1(s)|\cdot|\beta_2(t)|dsdt$$

$$\le \sup_{\|h\|_{p,q}\le 1} \int_J \left(\int_I |h(s,t)|^p ds\right)^{1/p}\cdot\|\beta_1\|_{p'}|\beta_2(t)|dt$$

$$\le \sup_{\|h\|_{p,q}\le 1} \|\beta_1\|_{p'}\|\beta_2\|_{q'}\left(\int_J\left(\int_I |h(s,t)|^p ds\right)^{q/p}dt\right)^{1/q} = 1.$$

Using product functions, we get $\|\sigma_{p,q}\| = 1$ for $1 \le p,q \le \infty$, hence condition (ii.a) of Lemma 1 is established.
Now we are going to check (ii.b), i. e. $\sigma(w) = 0$ for all blending functions. Indeed, let $w = \sum u_i g_i + \sum f_j v_j \in B(U,C(I);V,C(J))$. Then

$$\sigma(w) = \lambda\otimes\mu(w) = \sum \lambda(u_i)\mu(g_i) + \sum \lambda(f_j)\mu(v_j) = 0.$$

Finally, we prove (ii.c). By the definition of w^* we have $fg - w^* = (f - u^*)(g - v^*)$, and

$$E_U^p(f)E_V^q(g) = \lambda(f)\mu(g) = \sigma(fg) \le E^{p,q}_{B(U,C(I);V,C(J))}(fg)$$

$$\le \|fg-w^*\|_{p,q} = \|f-u^*\|_p\|g-v^*\|_q = E_U^p(f)E_V^q(g),$$

hence condition (ii.c). In particular, $\sigma(fg) = E^{p,q}_{B(U,C(I);V,C(J))}(fg)$.
Now Lemma 1 implies that w^* is a proximum to fg with respect to $B(U,C(I);V,C(J))$ and the norm α_{pq}. □

4. Mixed norm approximation in $\|.\|_{\infty,q}$. We consider the approximation with respect to an arbitrary subspace W of C(I×J) in the norm $\|.\|_{\infty,q}$, $1<q<\infty$. For q=1, a corresponding approximation problem was dealt with by Cheney-McCabe-Phillips [2] under modified assumptions.

Given an $h \in C(I\times J) \setminus \overline{W}$, then we ask whether a function $w^* \in W$ is a proximum to h, i. e. whether $h-w^*$ is minimal (over W) in the norm

$$\|h - w^*\|_{\infty,q} = \max_{x\in I}\|h_x - w_x^*\|_q = \max_{x\in K}\int_J (h-w^*)(x,t)\beta(x,t)dt,$$

where $\beta(x,t) := \operatorname{sign}(h(x,t)-w^*(x,t))\cdot|h(x,t)-w^*(x,t)|^{q-1}/\|h-w^*\|_q^{q-1}$ is given according to Section 2, and $K := K(h-w^*)$ is the set of all $x\in I$ for which the maximum is attained. Since ω is continuous (see Section 2) and I compact, we have $K \neq \emptyset$. We remark that β yields a norm representing linear functional φ with $\|\varphi\| = 1$, and that there is only one such functional for $h_x - w_x^* \neq 0$ (by the equality condition in Hölder's inequality, cf. Yosida [12, p. 34]).

Theorem 3.

Let W be a linear subspace of C(I×J), and $h \in C(I\times J)\setminus\overline{W}$. A function w^* in W is a proximum to h (with respect to $\|.\|_{\infty,q}$, $1<q<\infty$) iff

$$\min_{x\in K(h-w^*)}\int_J w(x,t)\beta(x,t)dt \le 0 \qquad \text{for all } w\in W.$$

Proof. The idea is that an eventual improvement of the approximation w^* is governed by the behavior on K. To make this more precise we follow Cheney-McCabe-Phillips [2]. Thus we introduce (for $1/q'+1/q=1$) $V := \{v \in L_{q'}(J) \mid \|v\|_{q'} \le 1\}$. By the Alaoglu-Bourbaki theorem, V is weak*-compact. For $g\in C(I\times J)$, define $\tilde{g}$ on I×V (endowed with the product topology) by $\tilde{g}(x,v) := \int_J v(t)g(x,t)dt$. Then we have for (x,v), $(x_o,v_o) \in X\times V$:

$$|\tilde{g}(x,v) - \tilde{g}(x_o,v_o)| \le \int_J |v(t)|\cdot|g(x,t)-g(x_o,t)|dt$$
$$\qquad + \left|\int_J v(t)g(x_o,t)dt - \int_J v_o(t)g(x_o,t)dt\right|$$
$$\le \|v\|_{q'}\|g_x - g_{x_o}\|_q + \left|\int_J v(t)g(x_o,t)dt - \int_J v_o(t)g(x_o,t)dt\right|.$$

Now, if (x,v) tends to (x_o,v_o) in X×V, then the right hand side tends to zero by the continuity of ω and the weak*-convergence in V, hence $\tilde{g}$ is continuous. Now for $h\in C(I\times J)$ and $w^*\in W$ we have

$$\|h-w^*\|_{\infty,q} = \max_{x\in K}\int_J (h-w^*)(x,t)\beta(x,t)dt = \max_{(x,v)\in X\times V}(\tilde{h}-\tilde{w}^*)(x,v).$$

Thus the original approximation problem is converted into a Chebyshev approximation problem, hence Kolmogorov's criterion applies (see Meinardus [8]) and yields our result. □

5. Remarks. Our result in Section 4 carries over to $C(I)\otimes_\alpha X$ instead of $C(I\times J)$, where α is an appropriate cross-norm (see Gilbert-Leih [5, especially p. 207]), X a normed vector space. But one has to be cautious if the norm representing functional is not uniquely determined (as possible in the case $\alpha_{\infty 1}$; see also Cheney-McCabe-Phillips [2]). In addition, the investigations of Section 3 can be extended to the case of spaces $L_p(I)\otimes_{\alpha_{pq}} L_q(J)$ for $1\le p,q<\infty$, and $C(I)\otimes_{\alpha_{\infty q}} L_q(J)$ or $L_p(I)\otimes_{\alpha_{p\infty}} C(J)$ rather than $C(I\times J)$, where the norms α_{pq} $(1\le p,q\le\infty)$ are extensions of the mixed norms in Section 3 to these larger spaces.

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