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THE GROWTH OF QUASISYMMETRIC FUNCTIONS

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1. Introduction. This lecture describes some joint work of the author and A. Hinkkanen [3, 4, 5]. An increasing homeomorphism $y = f(x)$ of the real axis $\mathbb{R}$ onto itself is called k quasisymmetric (k.q.s.) if the inequality

$$(1.1)\quad \frac{1}{k} \le \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \le k$$

holds for $x \in \mathbb{R}$ and $t > 0$. If f is k.q.s. for some k, $1 \le k < \infty$, then f is called q.s.

Beurling and Ahlfors [1] showed that the q.s. functions are precisely those which permit an extension to a quasiconformal homeomorphism of the plane onto itself. This justifies an independent study of the behaviour of these functions.

We can make a linear transformation of the x or y axes without affecting (1.1). Thus without loss of generality we may study the normalised class

$$N_0(k) = \{f \mid f(1) = 1,\ f(-1) = -1,\ f \text{ is k.q.s.}\}\ .$$

This turns out to be a compact family so that we can study bounds for $f(x)$ when x is fixed and also the behaviour of individual functions in $N_0(k)$.

Following Beurling and Ahlfors we define, when $\alpha > 0$

$$f_\alpha(x) = x^\alpha \ , \quad x \ge 0$$

$$f_\alpha(x) = -|x|^\alpha\ , \quad x < 0\ .$$

To test (1.1) it is evidently sufficient to take $x = 1$, so that we need only calculate the maximum and minimum values of $\{f_\alpha(1+t) - f_\alpha(1)\}/\{f_\alpha(1) - f_\alpha(1-t)\}$.

If $\alpha > 1$ the minimum value is 1 and the maximum is attained for $t = t_\alpha$, where t_α is the solution of

(1.2) $$(t+1)^{1-\alpha} + (t-1)^{1-\alpha} = 2 \quad .$$

The corresponding value of k is given by

(1.3) $$k = \frac{(t+1)^{\alpha} - 1}{1 + (t-1)^{\alpha}} = \left(\frac{t+1}{t-1}\right)^{\alpha-1} \quad .$$

By solving (1.2) and (1.3) for α as a function of k we obtain a value $\alpha = \alpha_1(k)$.

If $\alpha < 1$ then the maximum of the quotient in (1.1) is 1 and the minimum is obtained by again solving (1.2). So instead of (1.3) we must write

(1.4) $$\frac{1}{k} = \frac{(t+1)^{\alpha} - 1}{1 + (t-1)^{\alpha}} = \left(\frac{t+1}{t-1}\right)^{\alpha-1} \quad .$$

By solving (1.2) and (1.4) for given k we obtain a value $\alpha = \alpha_2(k)$. As Beurling and Ahlfors showed $f_{\alpha}(x)$ is k.q.s. for $\alpha_1 \le \alpha \le \alpha_2$. Also

(1.5) $$0 < \alpha_2(k) < 1 < \alpha_1(k) \quad .$$

It is not difficult to deduce various consequences from (1.2) to (1.4) [3] . We find that t increases from 1 to 2 as α increases from 0 to ∞ , and $t = \sqrt{2}$ corresponds to $\alpha = 1$. For large k we have the asymptotic relations

(1.6) $$\alpha_1(k) = \frac{\log k}{\log 3} + A + \frac{O(1)}{\log k} \quad , \quad \alpha_2(k) = \frac{2}{k \log 2} + O\left\{\frac{\log k}{k^2}\right\} \quad ,$$

where $A = 1 - (\log 4)/(\log 27)$.

<u>2. Bounds for</u> $f(x)$. We now discuss to what extent the functions $f_{\alpha_1}(x)$ and $f_{\alpha_2}(x)$ introduced by Beurling and Ahlfors [1] represent the extreme growth of k.q.s. functions. The first upper and lower bounds were obtained by Kelingos [6] . To save time we confine ourselves to upper bounds. The results for lower bounds are similar. Consider then

$$M(x\ , k) = \sup\{f \in N_o(k) \,|\, f(x)\} \quad .$$

We easily see by induction that

(2.1) $$M(3^n) \le (2k+1)^n \quad .$$

In fact (1.1) yields

$$f(3) - f(1) \le k\{f(1) - f(-1)\} \quad ,$$

which proves (2.1) for $n = 1$. If the result is known for n , then

$$f(3^{n+1}) - f(3^n) \le k\{f(3^n) - f(-3^n)\}$$

i.e.

$$f(3^{n+1}) \le (k+1)f(3^n)+k\{-f(-3^n)\} \le (2k+1)M(3^n) \le (2k+1)^{n+1} ,$$

since $-f(-x)$ also belongs to $N_0(k)$. This yields (2.1). We deduce that for $3^n < x \le 3^{n+1}$ we have

$$\log f(x) \le (n+1)\log(2k+1) \le n\log(2k+1) + \log(2k+1)$$

$$\le \frac{(\log x)\log(2k+1)}{\log 3} + \log(2k+1) ,$$

i.e.

$$(2.2) \qquad f(x) \le (2k+1)\, x^{\frac{\log(2k+1)}{\log 3}} .$$

The upper bound of Kelingos [6] was $f(x) \le (2x)^a$, where $a = \{\log(k+1)\}/\log 2$. From (2.2) we deduce that

$$(2.3) \qquad \alpha_1(k) \le \frac{\log(2k+1)}{\log 3} = \frac{\log k}{\log 3} + \frac{\log 2}{\log 3} + \frac{O(1)}{k} , \quad \text{as } k \to \infty .$$

However since strict inequality holds in (2.3) we still do not have the right order of magnitude for the exponent in (2.2).

We obtained (2.2) by taking $t = 2x$ in (1.1). To get the exact exponent $\alpha_1(k)$ we must choose $t = x t_1(k)$, where $t_1(k)$ is obtained by eliminating α from (1.2) and (1.3). With this value of t let f be a function in $N_0(k)$ for which $f\{x(1+t)\} = M\{x(1+t)\}$ is maximal. We deduce from (1.1)

$$(2.4) \qquad M\{x(1+t)\} = f(x+xt) \le (k+1)f(x) - k\,f(x-xt)$$

$$\le (k+1)M(x) + kM(tx-x) .$$

With this value of t and $\alpha = \alpha_1(k)$ we now set

$$(2.5) \qquad \psi(u) = \frac{M(e^u)}{e^{\alpha u}}$$

and

$$(2.6) \qquad u_1 = \log(t+1) , \quad u_2 = \log\left(\frac{t+1}{t-1}\right) .$$

Now (2.4), with $e^u = x + tx$, yields

$$(2.7) \qquad \psi(u) \le a\psi(u-u_1) + b\psi(u-u_2) ,$$

where

$$a = (k+1)e^{-\alpha u_1} = (k+1)(t+1)^{-\alpha} , \quad b = k\,e^{-\alpha u_2} = k\{(t-1)/(t+1)\}^{\alpha}$$

Using (1.2) and (1.3) we obtain

$$(2.8) \qquad a + b = \frac{2}{t+1} + \frac{t-1}{t+1} = 1 \quad .$$

Thus it remains to study the inequality (2.7), subject to (2.8). This is a special case of much more general convolution inequalities considered by Essén [2] and the results we need could be deduced from his. However in this simple case we can proceed directly. Since u_1 , u_2 are positive we need not restrict the behaviour of $\psi(u)$ as $u \to -\infty$ and we can prove directly that $\psi(u)$ is bounded as $u \to +\infty$.

We recall from section 1 that $\sqrt{2} < t < 2$, since $\alpha > 1$, so that

$$0 < u_1 < u_2 < \log \frac{\sqrt{2}+1}{\sqrt{2}-1} = \log(3 + 2\sqrt{2}) < \log 6 \ .$$

For any fixed u_o we set $u_n = u_o + n u_1$ and

$$(2.9) \qquad \mu_n = \sup_{u_n - u_2 \le u \le u_n} \psi(u) \quad .$$

Then μ_n decreases with increasing n . Suppose in fact that

$$u_n \le u < u_{n+1} \quad .$$

Then $u_n - u_2 \le u - u_2 < u - u_1 < u_{n+1} - u_1 = u_n$. Thus by (2.9)

$$\psi(u - u_1) \le \mu_n \quad , \quad \psi(u - u_2) \le \mu_n \quad ,$$

so that by (2.7)

$$\psi(u) \le \mu_n \quad .$$

In view of (2.9) this inequality also holds if $u_{n+1} - u_2 \le u < u_n$ and so

$$(2.10) \qquad \mu_{n+1} \le \mu_n \quad .$$

In particular if μ_o is finite then so is μ_n for $n \ge 1$, and $\mu_n \le \mu_o$. We also note for later use that μ_n tends to a limit μ as $n \to \infty$. At present we take $u_o = \log 3$. Then we have for $0 < u < u_o$

$$\psi(u) = \frac{M(e^u)}{e^{\alpha u}} < M(3) \le 2k + 1$$

by (2.1). For $-\log 3 < u < 0$ we have

$$\psi(u) = \frac{M(e^u)}{e^{\alpha u}} \le e^{\alpha \log 3} = 3^{\alpha} < 2k + 1$$

by (2.3). Since $u_2 < \log 6 < 2\log 3$, we deduce that $\mu_0 < 2k+1$. Thus

$$\mu_n < 2k+1 , \quad n \geq 0 ,$$

i.e.

$$(2.11) \qquad \psi(u) < 2k+1 , \quad u > -\log 3 .$$

We restate this as [3]

Theorem 1. We have

$$x^{\alpha_1} \leq M(x, k) < (2k+1)x^{\alpha_1} , \quad x > \frac{1}{3} .$$

The left hand inequality follows from the example $f_\alpha(x)$ of Beurling and Ahlfors. The right hand inequality is a consequence of (2.5), (2.11) and $\alpha = \alpha_1$.

3. Asymptotic behaviour. It is natural to ask whether $M(x, k)/x^\alpha$ tends to a limit as $x \to +\infty$, or whether $|f(x)|/|x|^\alpha$ tends to a limit as $x \to \infty$. It turns out that the answer to both questions is the same and depends on the arithmetic character of k. We have

Theorem 2 [4]. If $\log\{\frac{1}{2}(k+1)\}/\log k$ is irrational, then $M(x, k)/x^\alpha$ tends to a finite limit as $x \to +\infty$.
Also if $f \in N_0(k)$ then $|f(x)|/|x|^\alpha$ tends to a finite limit as $|x| \to \infty$.

Theorem 3. If $\log\{\frac{1}{2}(k+1)\}/\log k$ is rational then $M(x, k)/x^\alpha$ does not tend to a limit as $x \to +\infty$ [5] and there exist functions $f \in N_0(k)$ for which $|f(x)|/|x|^\alpha$ does not approach a limit as $|x| \to \infty$ [4].

We note that by (2.6) and (1.3)

$$u_2 = \frac{1}{\alpha - 1} \log k .$$

Also (1.2) and (1.3) give

$$\frac{k+1}{2} = \frac{1}{2}\left\{\left(\frac{t-1}{t+1}\right)^{1-\alpha} + 1\right\} = (t+1)^{\alpha-1} , \text{ so that } u_1 = \frac{1}{\alpha - 1} \log\left(\frac{k+1}{2}\right) .$$

Thus, with the hypotheses of Theorem 2, u_1/u_2 is irrational in (2.7). In this case we can show that $\psi(u)$ must tend to a finite limit. Let μ_n be the sequence satisfying (2.10) which tends to μ as $n \to \infty$. If $\mu = 0$ there is nothing to prove, since $\psi(u)$ then tends to 0. If $\mu > 0$, suppose that

$$\varliminf_{u \to \infty} \psi(u) = \lambda < \mu .$$

Choose $\eta = \frac{1}{4}(\mu - \lambda)$, $\lambda' = \lambda + 2\eta$. Then there exists a sequences v_n tending to ∞ with n , such that

$$\psi(v_n) < \lambda + \eta \ .$$

Choose δ so small that $(\lambda + \eta)e^{\alpha\delta} < \lambda + 2\eta$. Then for $v_n - \delta \leq v \leq v_n$, we have by (2.5)

$$\psi(u) = e^{-\alpha u}M(e^u) \leq e^{\alpha\delta} e^{-\alpha v_n}M(e^{v_n}) = e^{\alpha\delta}\psi(v_n) < \lambda' \ .$$

Since u_1/u_2 is irrational we can find a positive integer $N = N(\delta)$ such that the numbers $u + pu_1 + qu_2$ for $p > 0$, $q > 0$, $p + q \leq N$ and $v_n - \delta < u < v_n$ cover an interval I of length u_2 . With this value of N we choose $c = \min(a,b)$, $\varepsilon = c^N\eta$ and take n so large that

$$\psi(u) < \mu + \varepsilon \ , \quad u \geq v_n - \delta - u_2 \ .$$

Next we note that if $v_n - \delta \leq u \leq v_n$, then

$$\psi(u + u_1) \leq a\psi(u) + b\psi(u + u_1 - u_2) \leq a\lambda' + b(\mu + \varepsilon) \leq \mu + \varepsilon - c(\mu + \varepsilon - \lambda') \ .$$

Similarly we prove by induction on n , that if p , q are positive integers with $p + q \leq n$, then

$$\psi(u + pu_1 + qu_2) \leq \mu + \varepsilon - c^n(\mu + \varepsilon - \lambda') \ .$$

Choosing $n = N$, we deduce for u in I

$$\psi(u) < \mu + \varepsilon - c^N(\mu + \varepsilon - \lambda') < \mu + \varepsilon - 2c^N\eta = \mu - \varepsilon \ ,$$

and so, by the argument following (2.10), this inequality holds for all large u . This contradiction proves the first part of Theorem 2. We can prove the second part by applying the above argument with

$$e^{\alpha u}\,\psi_1(u) = \tfrac{1}{2}\{f(e^u) - f(-e^u)\} \quad \text{and} \quad e^{\alpha u}\,\psi_2(u) = \max\{f(e^u), -f(-e^u)\} \ .$$

Both $\psi = \psi_1$ and $\psi = \psi_2$ satisfy (2.7) and so ψ_j tends to a limit λ_j as $u \to +\infty$. We can then prove that $\lambda_1 = \lambda_2$ and complete the proof of Theorem 2.

In the rational case the situation is different. In this case if $u_2/u_1 = p/q$, a fraction in its largest terms, we define

$$h = \frac{1}{p(\alpha - 1)} \log\{\tfrac{1}{2}(k + 1)\} = \frac{1}{q(\alpha - 1)} \log k \ .$$

Hinkkanen [5] has constructed a piecewise linear function $f \in N_0(k)$ which is equal to $M_0(x, k)$ for sufficiently many x for the asymptotic behaviour to be determined. In the rational case this function $f(x)$ proves the statements of

Theorem 3. For large u the function $\psi(u)$ approximates a non constant periodic function of period h, which can be precisely calculated.

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