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A NEW ESTIMATE OF THE DEGREE OF MONOTONE INTERPOLATION

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1.Introduction

Given a set of real points $\{x_i, y_i\}_{i=0}^{n}$ (which we shall further refer to as data set), such that $x_0 < x_1 < \ldots < x_n$ and $y_0 < y_1 < \ldots < y_n$, there exists [1,2,3] an algebraic polynomial P with the properties:

i) $P(x_i) = y_i$, $i = 0, 1, \ldots, n$,

ii) $P(x)$ is monotone decreasing on $[x_{i-1}, x_i]$ if $y_{i-1} > y_i$ and monotone increasing on $[x_{i-1}, x_i]$ if $y_{i-1} < y_i$.

The polynomial P is called a partially monotone interpolation polynomial. In the situation when $y_{i-1} < y_i$, $i = 1, 2, \ldots, n$ P is called monotone interpolation polynomial (m.i.p.). The problem of estimating the degree of m.i.p. is recently considered by many authors, using certain characteristics of the data set. In what follows we shall restrict our considerations of the degree of m.i.p. First of all, we note that if $y_{i-2} < y_{i-1} = y_i$ at least for some i, then there exists no monotone increasing interpolation polynomial. In this situation we may assume that the degree of m.i.p. P is infinity ($\deg P = \infty$). This means that the characteristics

(1)
$$B = \min \{ \Delta y_i \ : \ i = 1, 2, \ldots, n \}$$

($\Delta y_i = y_i - y_{i-1}$) is essential for the adequate estimation of $\deg P$. Similarly, it turns out that the characteristics

(2)
$$A = \max \{ \Delta y_i \ : \ i = 1, 2, 3, \ldots, n \} ,$$
$$C = \min \{ \Delta x_i \ : \ i = 1, 2, 3, \ldots, n \}$$

are also important. All estimates in [4 - 11] make use of the characteristics A,B,C.

There are two approaches to the estimation of the degree of m.i.p.. The first one uses estimates for the uniform approximation of a monotone continuous or differentiable function by means of monotone algebraic polynomials. G.Lorentz and K.Zeller proved [8] that the order of approximation of a monotone function f by monotone algebraic polynomial of degree $\leqq n$ is $O(\omega(f;n^{-1}))$, where $\omega(f;\delta)$ is modulus of continuity of f. Later on R.Devor [12] proved, that the order of approximation of a monotone function f with k-th derivative by monotone algebraic polynomials of degree $\leqq n$ is $O(\omega(f^{(k)};n^{-1})n^{-k})$. Thus, making use of the result of [8], E.Passow and L.Raymon [4] prove the following estimate for the degree of m.i.p.

(3) $$\deg P = O\left(\frac{A}{BC}\right).$$

This estimate is exact in respect to the order in the situation when the ratio A/B is uniformly bounded with respect to n. Otherwise, this estimate implies in general much larger degrees of m.i.p.

The second approach makes use of the estimate for the Hausdorff approximation of the jump-function

$$f(x) = \begin{cases} -1, & -1 \leqq x \leqq 0, \\ 1, & 0 < x \leqq 1, \end{cases}$$

by means of monotone algebraic polynomials [13]. Namely, for every δ, $0 < \delta < 1/2$ there exists an odd monotone increasing algebraic polynomial P, such that:
i) $\deg P \leqq n$; ii) $0 < P(x) < 1$ if $x \in (0,\delta)$; iii) $1-\exp(-c_1 n\delta) \leqq P(x) \leqq 1$ for $x \in [\delta, 1]$, where $C_1>0$ is a constant independent of n and δ. This fact has been successfully used in the papers of M.Nikolceva [6]and G.Iliev [7]. It is proved in [6] that if the knots $\{x_i\}_{i=0}^n$ are equidistant and $A/B \asymp n^\alpha$, where $\alpha \geqq 1$, then

(4) $$\deg P = O(n \ln n).$$

In this case it follows immediately from (3) that $\deg P = O(n^{1+\alpha})$. The estimate (4) has been improved in [7] by

(5) $$\deg P = O\left(\frac{1}{C}\ln\left(\frac{A}{B}+e\right)\right).$$

The estimates (3) and (4) follow as spacial cases of (5). Moreover, (5) is order-exact for more general classes of data sets. However, there are data sets for which the estimate (5) is not order exact. For instance, if data set is $\left\{\frac{i}{n}, f_\alpha\left(\frac{i}{n}\right)\right\}_{i=0}^n$, where

$$f_\alpha(x) = |x-\tfrac{1}{2}|^\alpha \operatorname{sign}(x-\tfrac{1}{2}),\ x \in [0,1],\ 0<\alpha<1,$$

taking into account that $A/B \asymp n^{1-\alpha}$ and $C = n^{-1}$, then (5) implies $\deg P = O(n \ln n)$. On the other hand it is known [14], that for this data set we have $\deg P = O(n)$

Purpose of the present work is improve the estimate (5) in such a way that it will be order-exact for wider class of data sets, which will in particular include the above mentioned data set. To this end we shall first introduce an additional characteristics of the data set $\{x_i, y_i\}_{i=0}^{n}$, which is discrete analogue of the modulus of continuity of the function $\theta_f(x) = \operatorname{arctg} D(f; x)$ [15,16]

(6) $D = \max\{|\Delta\alpha_i|,\ i = 2,3,\dots,n\}$,

where $\alpha_i = \operatorname{arctg} \Delta y_i / \Delta x_i$, $i = 1,2,\dots,n$. Evidently α_i is the angle between the line passing through the points (x_{i-1}, y_{i-1}), (x_i, y_i) and the real axes.

2.Main result.

Theorem. Given a data set $\{x_i, y_i\}_{i=0}^{n}$ $(x_0 < x_1 < \dots < x_n,\ y_0 < y_1 < \dots < y_n)$ there exists an algebraic polynomial P and an absolute constant c_0, such that

i) $P(x_i) = y_i$, $i = 0,1,\dots,n$,

ii) $P'(x) \geqq 0$, $x \in [x_0, x_n]$,

iii) $\deg P \leqq c_0 \dfrac{x_n - x_0}{C} \ln\left(\dfrac{AD}{B} + \dfrac{AD}{C} + \dfrac{CD}{B} + e\right)$,

where A,B,C,D are the characteristics defined in (1),(2) and (6).

3. Auxiliary propositions.

Lemma 1. The system

$$t_i + \sum_{j=1}^{n} \varepsilon_{i,j} t_j = b_i,\quad i = 1,2,\dots,n,$$

where $\varepsilon_{i,j}$ are real numbers and $b_i > 0$, $i = 1,2,\dots,n$, has an unique positive solution, if there exist a monotone sequence of real numbers $x_0 < x_1 < \dots < x_n$ and $q > 0$, such that

i) $3q + |\varepsilon_{k,k}| \leqq 1$, $k = 1,2,\dots,n$,

ii) $\sum_j |\varepsilon_{k,j}| \Delta x_j \leqq qC$, $k = 1,2,\dots,n$,

iii) $\sum_j |\varepsilon_{k,j}||x_j - x_k| \Delta x_j \leqq q \dfrac{\min(C^3, BC^2)}{D(A+C)}$, $k = 1,2,\dots,n$,

where A,B,C,D are characteristics defined in (1),(2),(6) and $b_i = \Delta y_i$

Lemma 2. For any positive integers m and r (m – even, $r \geqq 2$) there exists a positive algebraic polynomial P of degree (m-1)r and monotone increasing in $[-1,1]$, such that for any $\delta\ (0 < \delta < 1)$ the following relations hold:

i) $\displaystyle\int_{-1}^{-\delta} |\sigma(x) - P(x)|\,dx = \int_{\delta}^{1} |\sigma(x) - P(x)|\,dx \leqq \delta \left(\frac{\pi^2}{2m\delta}\right)^{2r-1}$,

ii) $\displaystyle\int_{-1}^{-\delta} |x||\sigma(x) - P(x)|\,dx = \int_{\delta}^{1} x|\sigma(x) - P(x)|\,dx \leqq \delta^2 \left(\frac{\pi^2}{2m\delta}\right)^{2r-1}$,

iii) $1-P(x) = P(-x)$, $0 \leqq x \leqq 1$,

where $$\sigma(x) = \begin{cases} 0 , & x \in [-1,0] , \\ 1 , & x \in (0,1] . \end{cases}$$

Corollary 1. Let the data set $\{x_i , y_i\}_{i=0}^{n}$ be

$$x_i = i/n , y_i = f_\alpha(x_i) , i = 0,1,2, \ldots , n ,$$

where $f_\alpha(x) = |x-1/2|^\alpha \operatorname{sign}(x-1/2), x \in [0,1]$, and $0 < \alpha \leqq 1$.
Taking into account that $A = O(n^{-\alpha})$, $B = O(n^{-1})$, $C = n^{-1}$, $D = O(n^{\alpha-1})$, the Theorem implies of a m.i.p. P such that $\deg P = O(n)$.

Corollary 2. Let the data set $\{x_i , y_i\}_{i=0}^{n}$ be $x_i = i/n$, $y_i = (1+\ln x_i^{-1})^{-1}$, $i = 1,2, \ldots , n$ and $(x_0 , y_0) = (0,0)$. Taking into account that $A = O((\ln n)^{-1})$ $B = O(n^{-1})$, $C = n^{-1}$, $D = O(n^{-1}\ln^2 n)$ the Theorem implies of a m.i.p. P such that $\deg P = O(n \ln \ln n)$

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