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# APPROXIMATION BY BERNSTEIN POLYNOMIALS IN $L_p$ METRIC

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1. Main results. In this paper we consider the approximation of a function f defined in [0,1] by Bernstein polynomials

$$B_n(f;x)=\sum_{k=0}^{n} f(k/n)p_{k,n}(x); \qquad p_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}.$$

In 1912 S. Bernstein [5] introduced these polynomials and showed their uniform convergence to every continuous function. To approximate integrable (in the sense of Lebesgue) functions Kantorovich [12] considered the following modification

$$K_n(f;x)=\sum_{k=0}^{n} p_{k,n}(x)(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt.$$

Since then a lot of papers devoted to these operators have appeared. Here we shall list a few of so-called global results, i.e. results concerning the behaviour of the approximation on the whole interval [0,1] .

We denote by $\|.\|_p$ the $L_p$ norm $(1\leq p<\infty)$ and by $\|.\|_\infty$ the sup norm, but we make a difference between the spaces $L_\infty[0,1]$ and $C[0,1]$. We set $(1\leq p,q\leq \infty \ ,k\in \mathbb{N})$

$$\Delta_h^k f(x)=\begin{cases}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}f(x+mh), & \text{if } x,x+kh\in[0,1];\\ 0 & \text{otherwise,}\end{cases}$$

$$\omega_k(f,x;\delta) = \sup\left\{\left|\Delta_h^k f(t)\right| : \ t,t+kh\in[x-k\delta/2,x+k\delta/2]\right\},$$

$$\omega_k(f;\delta)_p = \sup\left\{\left\|\Delta_h^k f\right\|_p : |h|\leq\delta\right\} \quad \text{for } \delta=\text{const},$$

$$\tau_k(f;\delta)_p = \|\omega_k(f,\cdot,(\cdot))\|_p \qquad (\delta \text{ may be a function of } x),$$

$\Delta(t,x) = t\sqrt{x(1-x)} + t^2/2$, $\Delta_s(x) = \Delta(s^{-1},x)$, $\varphi(x) = x(1-x)$.

For the saturation class of Bernstein polynomials in C we have

Theorem A. For $f\in C[0,1]$ the following are equivalent:

(i) $\|f-B_nf\|_\infty = O(n^{-1})$ $(n\to\infty)$

(ii) $\tau_2(f,\Delta(t))_\infty = O(t^2)$ $(t\to 0+)$

(iii) $f\in AC$, $f'\in AC_{loc}$ and $\varphi f''\in L_\infty$.

Here $g\in AC_{loc}$ means that $g\in AC[a,b]$ for each closed subinterval $[a,b]$ of the open interval $[0,1]$.

In [14] Lorentz and Schumacher states that the saturation class for Bernstein polynomials is $\{f\in AC\colon f'\in AC \text{ and } \varphi f''\in L_\infty\}$. This is not so but actually their proof provides the implication (i)=>(iii) in the above theorem. The implication (ii)=>(i) follows from Theorem 1 in [11] (a Zygmund type theorem). At the end the implication (iii) =>(ii) is easily obtained by direct calculations.

In the non-saturation case we have

Theorem B. (Ditzian [7], Ivanov [11]) For $0<\theta<1$ and $f\in C[0,1]$ the following are equivalent:

(i) $\|f-B_nf\|_\infty = O(n^{-\theta})$ $(n\to\infty)$

(ii) $\tau_2(f,\Delta(t))_\infty = O(t^{2\theta})$ $(t\to 0+)$

(iii) $E_n(f)_\infty = O(n^{-2\theta})$ $(n\to\infty)$,

where $E_n(f)_\infty$ denotes the best uniform approximation of f by algebraic polynomials of n-th degree.

There are many other results concerning the weighted approximation or giving another structural characteristics of the statements (i) above (see e.g. Berens-Lorentz [4], Ditzian [6], Becker-Nessel [2]).

Now let us consider Kantorovich polynomials. We have in the saturation case

Theorem C. (Maier [16], Riemenschneider [18], Totik [20]) For $f\in L_p$ $(1<p<\infty)$ the following are equivalent:

(i) $\|f-K_nf\|_p = O(n^{-1})$ $(n\to\infty)$

(ii) $\|\Delta^2_{h\sqrt{\varphi}}f\|_p = O(h^2)$ $(h\to 0+)$

(iii) $f=g$ a.e. such that $g\in AC$, $g'\in L_p\cap AC_{loc}$ and $\varphi g''\in L_p$.

Theorem D. (Maier [15], Becker-Nessel [3]) For $f\in L_1$ the following are equivalent:

(i) $\|f - K_n f\|_1 = O(n^{-1}) \quad (n\to\infty)$

(ii) $\|\varphi\Delta_h^2 F\|_{BV} + \|\varphi\Delta_h^2 F\|_\infty = O(h^2) \quad (h\to 0+)$, where $F(x)=\int_0^x f(t)dt$

(iii) $f = g$ a.e. such that $g\in AC_{loc}$ and $\varphi g'\in BV$.

We have for the nonsaturation case

Theorem E. (Totik [19,20]) For $0<\theta<1$, $f\in L_p$ $(1\leq p<\infty)$ the following are equivalent:

(i) $\|f - K_n f\|_p = O(n^{-\theta}) \quad (n\to\infty)$

(ii) $\|\Delta^2_{h\sqrt{\varphi}} f\|_p + h^{\theta}\omega_1(f;h)_p = O(h^{2\theta}) \quad (h\to 0+)$

(iii) $E_n(f)_p = O(n^{-2\theta}) \quad (n\to\infty)$,

where $E_n(f)_p$ denotes the best $L_p$ approximation of $f$ by algebraic polynomials of n-th degree.

The common feature of all results cited above is that the operators ($B_n$ or $K_n$ ) are bounded (this is the reason why Kantorovich introduced its modification of Bernstein operators for $L[0,1]$). In this paper we consider Bernstein approximation in $L_p$ $(1\leq p<\infty)$, i.e. the operators $B_n$ are unbounded even for continuous functions. This is the main difficulty reflecting to the impossibility to get the full analog of the above theorems. As far as we know only two results in this field have appeared till now.

Theorem F. (Hoeffding [8]) If $f\in BV$, then

$$\|B_n f - f\|_1 \leq \sqrt{e/(2n)} \int_0^1 \sqrt{x(1-x)}\, df(x) + (n+1)^{-1} \overset{1}{\underset{0}{V}} f.$$

Theorem G. (Andreev-Popov [1]) If $1\leq p\leq\infty$, $f$ is bounded and measurable, then

$$\|f - B_n f\|_p = O(1)\, \tau_2(f;n^{-1/2})_p.$$

While Theorem F deals with a very special case, Theorem G treats the general case but without taking into consideration the improvement of the approximation near the end-points of the interval. As an immediate consequence of the properties of $\tau$ moduli and Theorem G we get that

$$\|f - B_n f\|_p \to 0 \quad (n\to\infty)$$

for each integrable in the sense of Riemann function $f$ $(1\leq p<\infty)$.

Now let us state our results.

Theorem 1. If $f$ is bounded and measurable, $1\leq p\leq\infty$, then

$$\|f - B_n f\|_p = O(1)\, \tau_2(f;\Delta_{\sqrt{n}})_p$$

Theorem 2. Let f be continuous, $1 < p < \infty$. Then the following are equivalent:

(i) $\|f - B_n f\|_p = O(n^{-1})$ $(n \to \infty)$

(ii) $\tau_2(f;\Delta(t))_p = O(t^2)$ $(t \to 0+)$

(iii) $f \in AC$, $f' \in AC_{loc} \cap L_p$ and $\varphi f'' \in L_p$.

We get as consequences of Theorem 1 both Theorem F (with an worse constant) and Theorem G. Theorem 2 shows that at least in the $O(n^{-1})$ case we can invert Theorem 1 if $p > 1$. Also the saturation classes for Bernstein operators in $L_p \cap C$ $(p > 1)$ are determined because Theorem 2 can be completed with

Theorem 3. Under the conditions of Theorem 2 we have

$$\|f - B_n f\|_p = o(n^{-1}) \text{ iff } f \text{ is linear.}$$

Let us also mention that from Theorem 1 can be derived the new result of V.Kostova [13] that for each continuous convex function f we have $\|f - B_n f\|_1 = O(n^{-1})$.

From Theorem C and Theorem 2 we get

Corollary. If $f \in C$, then the $L_p$ saturation classes of Bernstein and Kantorovich operators coincide $(1 < p < \infty)$.

Let us mention that this corollary is not true when $p=1$ or $p=\infty$ (cf. Theorem A and Theorem 4 below).

There are only two papers known to the autor determining the saturation classes of discrete-type operators in $L_p$ $(1 \leq p < \infty)$ - in [17] Popov and Szabados consider Jackson polynomials and its entire function analogs are consider by Dryanov (see the article in these Proceedings).

## 2. Proof of Theorem 1.

Following [4] we denote

$$(1) \quad S_u(x) = \begin{cases} u(x-1), & \text{if } 0 \leq u \leq x \leq 1; \\ x(u-1), & \text{if } 0 \leq x \leq u \leq 1. \end{cases}$$

Lemma 1 is a slight improvement of Lemma 3 in [4].

Lemma 1. Let $n \in \mathbb{N}$ and $g, g' \in AC[0,1]$. Then

$$B_n(g,x) - g(x) = \int_0^1 (B_n(S_u,x) - S_u(x))g''(u)du \quad \text{for } 0 \leq x \leq 1.$$

Proof. For x=0 or x=1 we obviously have 0=0. So let $0 < x < 1$. Then we have

$$(2) \quad \int_0^1 S_u(x)g''(u)du = -\int_0^x g'(u)du(x-1) - \int_x^1 g'(u)dx(u-1)$$

$= (1-x)(g(x)-g(0))-x(g(1)-g(x)) = g(x)-(1-x)g(0)-xg(1).$

Using (2) we get

(3) $$\int_0^1 B_n(S_u,x)g''(u)du = \sum_{k=0}^{n} p_{k,n}(x) \int_0^1 S_u(k/n)g''(u)du$$

$$= \sum_{k=0}^{n} p_{k,n}(x)(g(k/n)-(1-k/n)g(0)-k/ng(1))$$

$$= B_n(g,x) - (1-x)g(0) - xg(1),$$

because $B_n$ preserve the linear functions. Now (2) and (3) prove the lemma.

Some properties of the kernel from Lemma 1 are collected in

Lemma 2.

(4) $$B_n(S_u,x) - S_u(x) \geqq 0 \qquad \text{for } 0 \leqq x,u \leqq 1;$$

(5) $$\int_0^1 (B_n(S_u,x) - S_u(x))\varphi^{-1}(u)du \leqq 7n^{-1} \qquad \text{for } 0 \leqq x \leqq 1;$$

(6) $$\int_0^1 (B_n(S_u,x) - S_u(x))dx \leqq \varphi(u)n^{-1} \qquad \text{for } 0 \leqq u \leqq 1.$$

Proof. (4) and (5) are proved in [4]. From (1) we get

(7) $$\int_0^1 S_u(x)dx = -\varphi(u)/2$$

and

(8) $$(n+1)\int_0^1 B_n(S_u,x)dx = \sum_{k=0}^{n} S_u(k/n) = \sum_{k=0}^{[nu]}(u-1)k/n + \sum_{k=[nu]+1}^{n} u(k/n-1)$$

$$= (u-1)n^{-1}[nu]([nu]+1)/2 - un^{-1}(n-[nu]-1)(n-[nu])/2$$

$$= -n\varphi(u)/2 + \{nu\}(1-\{nu\})/(2n),$$

where $[y]$ and $\{y\}$ denote the integer and the fractional part of the real number $y$. Now (7) and (8) give

$$\int_0^1 (B_n(S_u,x) - S_u(x))dx = \varphi(u)/(2(n+1))$$

$$+\{nu\}(1-\{nu\})/(2n(n+1)) \leqq \varphi(u)/(n+1).$$

Lemma 3. For $n \in \mathbb{N}$ and $g \in AC$ such that $g' \in AC$ and $g'' \in L_p[0,1]$ $(1 \leqq p \leqq \infty)$ we have

$$\|B_n g - g\|_p \leqq 7n^{-1}\|\varphi g''\|_p.$$

Proof. We consider the linear operator

$$Tf(x) = \int_0^1 (B_n(S_u,x) - S_u(x))\varphi^{-1}(u)f(u)du.$$

From Lemma 2 we get

$$\|Tf\|_1 \leq \int_0^1\int_0^1 (B_n(S_u,x)-S_u(x))dx\, \varphi^{-1}(u)|f(u)|\,du \leq n^{-1}\|f\|_1 \quad \text{and}$$

$$\|Tf\|_\infty \leq \|f\|_\infty \cdot \sup\left\{\int_0^1 (B_n(S_u,x)-S_u(x))\varphi^{-1}(u)du \ : \ x\in[0,1]\right\}$$

$$\leq 7n^{-1}\|f\|_\infty .$$

Now Riesz-Thorin theorem gives $\|Tf\|_p \leq 7n^{-1}\|f\|_p$. This inequality with $f=\varphi g''$ and Lemma 1 prove the lemma.

Lemma 4. If $m=[\sqrt{n}]+1$ and $x\in[k/(n+1),(k+1)/(n+1)]$ for some $k = 0,1,\dots,n$ then $[x-26\Delta_m(x),x+26\Delta_m(x)]\supset[k/n-\Delta_m(k/n),k/n+\Delta_m(k/n)]$.

Proof. Let z be such that $|z-k/n|\leq\Delta_m(k/n)$. Then $|x-z|\leq|x-k/n|$ $+\Delta_m(k/n)$. But $|x-k/n|\leq n^{-1}\leq 4m^{-2}\leq 8\Delta_m(x)$. Now from (2.5) in [9] we get $\Delta_m(k/n)\leq 18\Delta_m(x)$, which proves the lemma.

We are ready to prove Theorem 1. We set $m=[\sqrt{n}]+1$. Theorem 3.1 in [10] restated for the interval [0,1] provides (k=2,w=1) a function $G_m\in W_p^2[0,1]$ such that

(9) $\quad |G_m(x) - f(x)| \leq c_1\,\omega_2(f,x;\Delta_m(x))$ and

(10) $\quad \|(\Delta_m)^2 G_m''\|_p \leq c_2\,\tau_2(f;\Delta_m)_p$.

Now Jensen inequality, (9) and Lemma 4 give

(11) $$\|B_nf - B_nG_m\|_p \leq \left\{\sum_{k=0}^{n} |f(k/n)-G_m(k/n)|^p \int_0^1 p_{k,n}(x)dx\right\}^{1/p}$$

$$\leq c_1\left\{(n+1)^{-1}\sum_{k=0}^{n}\omega_2^p(f,k/n;\Delta_m(k/n))\right\}^{1/p}$$

$$\leq c_1\left\{\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)} \omega_2^p(f,x;26\Delta_m(x))\,dx\right\}^{1/p}$$

$$= c_1\,\tau_2(f;26\Delta_m)_p \leq c_3\,\tau_2(f;\Delta_m)_p.$$

We can take the constant 26 out of the modulus by the following way. We represent the second finite difference with a step h as a linear combination of second finite differences with a step $hr^{-1}$ (r is natural and depends only on 26) and after that we evaluate $\tau_2(f;26\Delta(t))_p$

with $O(\tau_2(f;\Delta(t))_p)$.

Using Lemma 3 and (10) we get

(12) $$\|G_m - B_nG_m\|_p \le 7n^{-1}\|\varphi G_m''\|_p \le 28\|(\Delta_m)^2 G_m''\|_p \le 28c_2\,\tau_2(f;\Delta_m)_p.$$

At the end from (9), (12) and (11) we get

$$\|f - B_nf\|_p \le \|f - G_m\|_p + \|G_m - B_nG_m\|_p + \|B_nG_m - B_nf\|_p$$
$$\le c_4\,\tau_2(f;\Delta_m)_p \le c_4\,\tau_2(f;\Delta_{\sqrt{n}})_p.$$

3. Remarks and generalizations.

1. The proof of Theorem 2 and Theorem 3 will appear elsewhere. The most difficult part of this proof is of course the implication (i)=>(iii). The reason for this is the unboundness of $B_n$ which embarrasses the employment of the information provided by (i).

2. We do not determine the saturation class of Bernstein polynomials in $L_1$ because our method for proving inverse results does not work in this case.

3. The following theorem determines the saturation class of Kantorovich polynomials in $C[0,1]$. The result should be known but the autor can not find any references.

Theorem 4. Let $f \in C[0,1]$. Then the folloing are equivalent:

(i) $\|f - K_nf\|_\infty = O(n^{-1})$ $(n \to \infty)$

(ii) $f \in AC$, $f' \in AC_{loc} \cap L_\infty$ and $\varphi f'' \in L_\infty$.

Proof. The implication (ii)=>(i) follows from Theorem 1 in 18 . To get the inverse implication we use Theorem 4.3 in [14]. Then we have $f' \in AC_{loc}$ and $\operatorname{ess\,sup} |g'(x)| = M < \infty$, where $g(x) = x(1-x)f'(x)/2$. If we assume that $g(0) \ne 0$ (or $g(1) \ne 0$), then f will be unbounded which contradicts with $f \in C$. So $g(0) = g(1) = 0$ and $|g(x)| \le 2Mx(1-x)$. Hence $|f'| \le 4M$ and $\operatorname{ess\,sup}|\varphi f''| \le 6M$.

The implication "$\varphi f'' \in L_\infty$" => "$f' \in L_\infty$" is not valid as the function $x.\log(x)$ shows. This function belongs to the saturation class of Bernstein polynomials in C but does not belong to the saturation class of Kantorovich polynomials.

4. Results similar to the results of this paper can be formulate for other discrete-type operators. The methods for proving the direct and inverse parts deal with common properties of the approximating processes and can be easily applied to variety of operators. For example we do not need the operators to preserve the polynomials of high degree. This property of Jackson polynomials plays a basic

role in the proof of the inverse theorem in [17].

5. Let us briefly discuss the non-saturation case. By analogy with Theorem B and Theorem 1 we may assume that the implication

$$"\|f-B_nf\|_p = O(n^{-\theta})" \Rightarrow "\tau_2(f;\Delta(t))_p = O(t^{2\theta})"$$

is true for each $0<\theta<1$ when $p<\infty$. This assertion is true when $\theta > 1/p$ but there are examples showing it is not true for little $\theta$. We do not know the exact lower bound of these $\theta$ for which the above asertion is valid.

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