КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 430—439

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
Sofia, 1984, p. 430-439

HERMITE SPLINE INTERPOLATION OF DATA OF POWER GROWTH

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1. Introduction. The Hermite Interpolation Problem on a prescribed linear function space $\mathcal{L}$ at a bi-infinite knot sequence $(z_i)_{i\in\mathbb{Z}}$, is as follows:

Given $r \in \mathbb{Z}_{++} := \{1,2,\ldots\}$, a sequence of pairs (z_i, μ_i) with $z_i < z_{i+1}$ and $\mu_i \in \{1,\ldots,r\}$ $(\forall\ i \in \mathbb{Z})$, and a data sequence (v_{ij}) $(j = 0,1,\ldots,\mu_i-1;\ i \in \mathbb{Z})$, does there exist a function $F \in \mathcal{L}$ such that

$$F^{(j)}(z_i) = v_{ij} \quad (j = 0,1,\ldots,\mu_i-1;\ \forall i \in \mathbb{Z})\ ?$$

Thus the knot z_i has multiplicity μ_i, and we have to match the function and its derivatives (up to order μ_i-1) at z_i to the specified data.

In our notation we shall stretch out the pairs (z_i, μ_i) into a single sequence of the form

$$\ldots,z_{i-1},\overbrace{z_i,z_i,\ldots,z_i}^{\mu_i \text{ times}},z_{i+1},\ldots\ ;$$

we shall denote this sequence by $x = (x_i)_{i\in\mathbb{Z}}$ and suppose that

$$x_i \le x_{i+1} \text{ and } x_i < x_{i+r} \quad (\forall\ i \in \mathbb{Z}), \qquad x_i \to \pm\infty \quad (i \to \pm\infty).$$

For each $i \in \mathbb{Z}$ we write

(1.1) $$i^- := \min\{\, j \mid x_j = x_i \,\}, \quad i^+ := \max\{\, j \mid x_j = x_i \,\};$$

thus whenever $i^- \le i \le i^+$ we have $x_{i^--1} < x_{i^-} = x_i = x_{i^+} < x_{i^++1}$, so that $\mu_i = i^+ - i^- + 1$ is the exact multiplicity of the knot at x_i (for a simple knot x_i, we have $i^- = i^+$). We shall suppose that r is the actual maximum knot multiplicity, namely $r = \max_{i\in\mathbb{Z}} (i^+ - i^- + 1)$.

Our data sequence (which may be real- or complex-valued) is now specified by a single bi-infinite sequence $y = (y_i)_{i\in\mathbb{Z}}$, and the Hermite Interpolation Problem HIP$(y, \mathcal{L}, x)$ then investigates the existence of $F \in \mathcal{L}$ such that

$$F^{(i-i^-)}(x_i) = y_i \quad (\forall\ i \in \mathbb{Z}).$$

The space $\mathcal{S}_{m,x}$ of spline functions of degree m-1 (or order m, where $m \geq r$) with knots x may be defined in terms of the above corresponding sequence of pairs (z_i, μ_i) as

(1.2) $\mathcal{S}_{m,x} := \{S(\cdot) \mid S \in C^{m-1-\mu_i}(z_{i-1}, z_{i+1}) \text{ and } S|_{[(z_i, z_{i+1}]} \in \pi_{m-1} \ (\forall\, i \in \mathbb{Z})\}$

where π_{m-1} denotes the set of all polynomials of degree not exceeding m-1, and the continuity condition involving $C^{m-1-\mu_i}$ is to be omitted if $\mu_i = r = m$.

The class of sequences of power growth $\rho \in \mathbb{R}_+ := \{t \mid t \geq 0\}$, will be denoted by

(1.3) $$\mathfrak{Y}_{\rho,x} := \{ y \mid y_i = O(|x_i|^\rho) \text{ as } |i| \to \infty \}$$

and the class of functions of power growth (where $f : \mathbb{R} \to \mathbb{C}$) by

(1.4) $$\mathfrak{P} := \{ f(\cdot) \mid \exists\, \sigma \in \mathbb{R}_+, f(t) = O(|t|^\sigma) \text{ as } |t| \to \infty \} .$$

Given a data sequence of power growth, $y \in \mathfrak{Y}_{\rho,x}$, it is our purpose here to investigate the existence, uniqueness, and representation of an odd-degree spline function of power growth which will solve the corresponding Hermite Interpolation Problem $\text{HIP}(y, \mathcal{S}_{2m,x} \cap \mathfrak{P}, x)$.

Lipow and Schoenberg [5] raised and solved this problem in the cardinal case $x_i := i$, where each knot is of the same fixed multiplicity r $(r \leq m)$, and for any $\rho \in \mathbb{R}_+$. For the case of simple knots $(r = 1)$ we may refer to Schoenberg [6], de Boor [1], Jakimovski-Russell-Stieglitz [3] ; although in that case it is not a "Hermite" interpolation problem, the results of the present paper reduce to those of [3] when $r = 1$.

2. Conditions on the knots.

Let $m \in \mathbb{Z}_{++}$. We suppose throughout that $x = (x_i)$ is a knot sequence as in §1, with maximum multiplicity $r := \max\, (i^+ - i^- + 1) \leq m$, and such that

(2.1) $$x_i \leq x_{i+1} \quad \text{and} \quad x_i < x_{i+r} \leq x_{i+m} \ (\forall\, i \in \mathbb{Z}), \quad x_i \to \pm\infty \quad (i \to \pm\infty) .$$

The following condition on the growth and smoothness of x will be used:

(2.2) given some $\varepsilon > 0$, $\exists\, \kappa_\varepsilon > 0$ such that

$$\forall\, i,j \in \mathbb{Z}, \quad \frac{x_{i^+ +1} - x_{i^+}}{x_{j^+ +1} - x_{j^+}} \leq \kappa_\varepsilon\, e^{\varepsilon |i-j|} .$$

We shall denote by $\mathfrak{X}$ the set of sequences x such that

(2.3) $$\mathfrak{X} := \{ x = (x_i) \mid x \text{ satisfies (2.1), and (2.2) for every } \varepsilon > 0 \}.$$

If x satisfies (2.1), and (2.2) for some $\varepsilon > 0$, we obviously have

(2.4) $$1 \leq \mu := \sup_{|i-j| = 1} \frac{x_{i^+ +1} - x_{i^+}}{x_{j^+ +1} - x_{j^+}} < \infty$$

(i.e., the local mesh ratio μ is finite); and we can also verify that

$$\frac{x_{i+m} - x_i}{x_{j+m} - x_j} \le c_\varepsilon\, e^{\varepsilon|i-j|} \qquad (\forall\ i,j), \tag{2.5}$$

$$\frac{1 + |x_i|}{1 + |x_j|} \le c'_\varepsilon\, e^{\varepsilon|i-j|} \qquad (\forall\ i,j). \tag{2.6}$$

The local mesh ratio property (2.4) also implies that

$$x_{k+m} - x_k \le c(1 + |x_k|)$$

and hence, since also $0 \le k-k^- \le r-1$,

$$\exists\, \alpha \in [0,r-1] \text{ such that } (x_{k+m} - x_k)^{k-k^-} \le c(1 + |x_k|)^\alpha \quad (\forall\ k \in \mathbb{Z}). \tag{2.7}$$

Here, and elsewhere throughout the paper, constants c are not necessarily the same at each occurrence.

Remark 1. If the global mesh ratio is finite, as for instance with cardinal knots, we may take $\alpha = 0$ in (2.7); or for simple knots we have $r = 1$ and hence $\alpha = 0$. But in any case we can replace α by $r-1$ in (2.7) if all we require is an upper estimate.

3. Fundamental splines.

For each $k \in \mathbb{Z}$ and $k \le i \le k+m$, and for x satisfying (2.1), denote

$$i_k^- := \max\,(i^-, k), \qquad i_k^+ := \min\,(i^+, k+m);$$

thus for a particular k, $i_k^+ - i_k^- + 1$ measures the relative multiplicity of the knot at x_i when i is restricted to the interval $[k,k+m]$. The divided difference $F[x_k,\ldots,x_{k+m}]$ of a function $F \in C^{m-2}[x_k,x_{k+m}]$ may now be expressed in the form (see Jakimovski-Stieglitz [4])

$$F[x_k,\ldots,x_{k+m}] = \sum_{i=k}^{k+m} \frac{1}{(i-i_k^-)!\,(i_k^+-i)!} \left(\frac{d}{dv}\right)^{i_k^+-i} \left(\frac{(v-x_i)^{i_k^+-i_k^-+1}}{(v-x_k)\ldots(v-x_{k+m})}\right)\Bigg|_{v=x_i} \cdot F^{(i-i_k^-)}(x_i). \tag{3.1}$$

If $y_i = F^{(i-i^-)}(x_i)$ $(\forall\ i)$ then we write $y[x_k,\ldots,x_{k+m}] = F[x_k,\ldots,x_{k+m}]$.

The definitions are consistent with the usual definitions of $y[x_k,\ldots,x_{k+m}]$ and $F[x_k,\ldots,x_{k+m}]$ for multiple knots, and with the standard definition of a divided difference when all the knots are simple. Moreover, the definition of B-splines may be expressed in the same way as for simple knots, in terms of divided differences of the function $F(u) = (u-t)_+^{m-1}$; more precisely, we define the normalized B-splines, for $k \in \mathbb{Z}$, $t \in \mathbb{R}$, by

$$N_{k,m}(t) := N_{k,m}(t;x) := (x_{k+m} - x_k)\,(\cdot-t)_+^{m-1}[x_k,\ldots,x_{k+m}]. \tag{3.2}$$

The Peano formula then continues to be valid for multiple knots, namely for $F^{(m-1)} \in AC[x_k,x_{k+m}]$ we have

$$(3.3)\qquad F[x_k,\dots,x_{k+m}] = \{(m-1)!(x_{k+m}-x_k)\}^{-1} \int_{x_k}^{x_{k+m}} F^{(m)}(t)\, N_{k,m}(t)\, dt.$$

The sequence space $\ell^m_{2,x}$ is now defined by

$$(3.4)\qquad \ell^m_{2,x} := \{\, y \mid \|y\|_{\ell^m_{2,x}} := (m^{-1} \sum_i (x_{i+m}-x_i)\, |y[x_i,\dots,x_{i+m}]|^2)^{\frac{1}{2}} < \infty \,\}.$$

Also $\mathfrak{L}_2 := \mathfrak{L}_2(\mathbb{R})$ denotes the Lebesgue function space with the usual norm, and the unit sequences $e^{(k)}$ $(k \in \mathbb{Z})$ satisfy

$$e^{(k)}_k = 1\,, \quad e^{(k)}_i = 0 \; (i \neq k).$$

For the following Lemma see Jakimovski-Stieglitz [4, Theorem 4] (the corresponding result for simple knots can be found in [2, Theorems 8(a),9] or [3, Lemma A]).

LEMMA A. <u>Let</u> $m \in \mathbb{Z}_{++}$ <u>and</u> x <u>satisfy</u> (2.1). <u>If</u> $y \in \ell^m_{2,x}$ <u>then there is a unique spline function</u> $S := S_y(\cdot,x)$ <u>with the properties</u>

$$S \in \mathfrak{S}_{2m,x}\,, \qquad S^{(m)} \in \mathfrak{L}_2\,, \qquad S^{(i-i^-)}(x_i) = y_i \quad (\forall\, i \in \mathbb{Z}).$$

<u>Moreover</u>, $\exists\, c_m$ <u>such that</u>

$$\|S^{(m)}\|_{\mathfrak{L}_2} \le c_m \|y\|_{\ell^m_{2,x}}\,.$$

Now for each k, $e^{(k)} \in \ell^m_{2,x}$ (the series in (3.4), which defines its norm, is finite). Thus given $m \in \mathbb{Z}_{++}$ and x satisfying (2.1), we may take $y = e^{(k)}$ in Lemma A and obtain, for each $k \in \mathbb{Z}$, the existence of a unique spline $L_k(\cdot) := L_{k,2m}(\cdot) := L_{k,2m}(\cdot\,;x)$ such that

$$(3.5)\qquad L_k \in \mathfrak{S}_{2m,x}\,, \qquad L_k^{(m)} \in \mathfrak{L}_2\,, \qquad L_k^{(i-i^-)}(x_i) = e_i^{(k)} \quad (\forall\, i \in \mathbb{Z}),$$

$$(3.6)\qquad \|L_k^{(m)}\|_{\mathfrak{L}_2} \le c_m \|e^{(k)}\|_{\ell^m_{2,x}}\,.$$

Following the nomenclature of Schoenberg [6, p.408] , de Boor [1, Lemma 1] , and Jakimovski-Russell-Stieglitz [3, §1] , all for the case of simple knots, and of Lipow-Schoenberg [5, §7] for the cardinal Hermite case, we call the above L_k <u>fundamental splines</u>.

<u>4. The main result.</u> Subject to the notation and conditions which we have introduced above, we may now state our principal result:

THEOREM. <u>Let</u> $m \in \mathbb{Z}_{++}$, $\rho \in \mathbb{R}_+$, <u>and let</u> x <u>be a bi-infinite sequence of knots of maximum multiplicity</u> r $(r \le m)$ <u>satisfying</u> $x \in \mathfrak{X}$ <u>(see</u> (2.3)). <u>Let</u> $i^- := \min\{j \mid x_j = x_i\}$ <u>and let</u> $\mathfrak{Y}_{\rho,x}$, $\mathfrak{P}$ <u>be defined as in</u> (1.3) <u>and</u> (1.4). <u>If</u> $y \in \mathfrak{Y}_{\rho,x}$ <u>then there is one and only one</u> $S(\cdot)$ <u>satisfying</u>

$$(4.1)\qquad S \in \mathfrak{S}_{2m,x} \cap \mathfrak{P}\,, \qquad S^{(i-i^-)}(x_i) = y_i \quad (\forall\, i \in \mathbb{Z}).$$

The spline S and its derivatives $S^{(q)}$ have the representations

$$(4.2)\qquad S^{(q)}(t) = \sum_{k\in\mathbb{Z}} y_k\, L^{(q)}_{k,2m}(t) \qquad (\forall\ t \in \mathbb{R};\ q = 0,1,\dots,m-1),$$

where L_k is given by (3.5) and each series in (4.2) converges uniformly on any compact subset of $\mathbb{R}$.

Moreover, if α is chosen as in (2.7) then, for $q = 0,1,\dots,m-1$,

$$(4.3)\qquad (x_{j+m} - x_j)^q\, S^{(q)}(t) = O(\,|t|^{\rho+\alpha}\,),\quad x_j \le t \le x_{j+m}\ ,\ |j| \to \infty\ .$$

We shall deduce from the following Lemmas B and C the existence and uniqueness of the spline solution S in this Theorem, but we postpone until §5 the proofs of Lemmas B and C.

Definition. $S \in \mathcal{S}_{2m,x}$ is called a null-spline when $S^{(i-i^-)}(x_i) = 0$ $(\forall\ i \in \mathbb{Z})$; it is non-trivial if $S(t) \neq 0$ for at least one t.

LEMMA B. If $m \in \mathbb{Z}_{++}$, $x \in \mathfrak{X}$, then $\mathcal{S}_{2m,x} \cap \mathfrak{P}$ contains no non-trivial null-spline.

LEMMA C. Let $m \in \mathbb{Z}_{++}$, $x \in \mathfrak{X}$. Then $\exists\, \lambda_1 = \lambda_1(m,x) > 0$ and $\exists\, c = c(m,x) > 0$ such that, for $x_{j^+} \le t \le x_{j^++1}$, $\forall\ j \in \mathbb{Z}$, $\forall\ k \in \mathbb{Z}$,

$$(4.4)\quad (x_{j+m}-x_j)^q\, |L^{(q)}_{k,2m}(t)| \le c\, (x_{k+m}-x_k)^{k-k^-} \cdot e^{-\lambda_1|k-j|} \quad (q = 0,1,\dots,m-1).$$

Proof of the Theorem.

Lemma B ensures that the HIP has at most one solution in $\mathcal{S}_{2m,x} \cap \mathfrak{P}$ for a given data sequence y. For suppose there were splines $S_1, S_2 \in \mathcal{S}_{2m,x} \cap \mathfrak{P}$ such that $S_1^{i-i^-}(x_i) = S_2^{i-i^-}(x_i) = y_i$ $(\forall\ i)$. Then $S := S_1 - S_2$ would be a null-spline in $\mathcal{S}_{2m,x} \cap \mathfrak{P}$ and hence, by Lemma B, S must be trivial, namely $S_1 = S_2$.

Lemma C ensures that the HIP has at least one spline solution of power growth (i.e., in $\mathcal{S}_{2m,x} \cap \mathfrak{P}$) for a given data sequence of power growth, by showing that $S(t) := \Sigma_k\, y_k\, L_k(t)$ has all the required properties. Thus consider the series in (4.2) : we have by Lemma C, for $x_{j^+} \le t \le x_{j^++1}$ and $y \in \mathfrak{Y}_{\rho,x}$,

$$(x_{j+m}-x_j)^q \sum_{k\in\mathbb{Z}} |y_k|\,|L_k^{(q)}(t)| \le c \sum_{k\in\mathbb{Z}} (1 + |x_k|)^\rho (x_{k+m}-x_k)^{k-k^-}\, e^{-\lambda_1|k-j|}$$

$$\le c \sum_{k\in\mathbb{Z}} (1 + |x_k|)^{\rho+\alpha}\, e^{-\lambda_1|k-j|} \qquad \text{by (2.7)}$$

$$\le c\, (1 + |x_j|)^{\rho+\alpha} \sum_{k\in\mathbb{Z}} e^{\varepsilon(\rho+\alpha)|k-j|}\, e^{-\lambda_1|k-j|} \qquad \text{by (2.6)}$$

$$(4.5)\qquad \le c\, (1 + |t|)^{\rho+\alpha} \sum_{p\in\mathbb{Z}} \theta^{|p|}\ ,$$

where we choose ε so small that $0 < e^{-[\lambda_1 - \varepsilon(\rho+\alpha)]} =: \theta < 1$. In the last step we also used the local mesh ratio property (2.4) to replace $|x_j|$ by $|t|$, and we can also use the same property to extend the values of t to the more conveniently expressed interval $[x_j, x_{j+m}]$. Hence the series (4.2) converge locally uniformly. Thus we may define

$$S(t) := \sum_{k\in\mathbb{Z}} y_k L_{k,2m}(t) \qquad (\forall\, t \in \mathbb{R}) \tag{4.6}$$

and differentiate term by term to obtain the representations (4.2).

It is clear, from the definition of the L_k in (3.5), that if we take $t = x_i$ and $q = i-i^-$ in (4.2), we get $S^{i-i^-}(x_i) = y_i$. Since $L_k \in \mathcal{S}_{2m,x}$ for each k, it follows from (4.2) that $S \in \mathcal{S}_{2m,x}$. Moreover, combining (4.2) and (4.5), we get the estimates (4.3). The case $q = 0$ of (4.3) shows that $S \in \mathfrak{P}$. Consequently S has all the properties specified, and the proof is complete. □

<ins>Remark 2.</ins> In our Theorem, and in Lemmas B and C, we have stated our hypotheses on the mesh ratio in the convenient form that x should satisfy (2.2) for <ins>every</ins> $\varepsilon > 0$. However, throughout this paper, whenever we use this fact we need only to be able to pick <ins>some suitable small fixed</ins> $\varepsilon > 0$. For example, in (4.5) we needed to pick ε so that $\lambda_1 - \varepsilon(\rho+\alpha) > 0$, but since λ_1 is difficult to determine, we were unable to specify ε precisely. Compare [3, Remark 1] .

<ins>Remark 3.</ins> While in Lipow and Schoenberg's treatment [5] of the <ins>cardinal</ins> Hermite interpolation problem, they are able to make the interpolating spline S and all its derivatives $S^{(q)}$ up to $q = r-1$ have the same order of power growth ρ as the data sequence, the best we have obtained in the non-cardinal case is (4.3). Obviously in the cardinal case, $\alpha = 0$ and $x_{j+m} - x_j = m$ and then (4.3) does indeed give us $S^{(q)}(t) = O(|t|^\rho)$ $(0 \le q < m)$. But while in the non-cardinal case we certainly get $S \in \mathfrak{P}$ as we require (take $q = 0$ in (4.3)), we cannot expect (except when $\inf_{j\in\mathbb{Z}}(x_{j+m} - x_j) > 0$) to necessarily get $S^{(q)} \in \mathfrak{P}$ for some $q > 0$. For example, take

$$x_{2j} = x_{2j+1} = \log(1 + |j|)\cdot\operatorname{sgn} j \qquad (\forall\, j \in \mathbb{Z}).$$

Then $x \in \mathfrak{X}$ as required in the Theorem, and $r = 2$. We can take $\alpha = 0$ in (2.7) because $x_{k+m} - x_k$ is bounded above, but the best we can do from (4.3) is now

$$S^{(q)}(t) = O(|t|^\rho e^{q|t|}) \quad (0 \le q < m).$$

<ins>5. Proof of Lemmas B and C.</ins>

<ins>Proof of Lemma B.</ins> The special case of this lemma when all the knots are simple, has already been proved in [3, Lemma 2] . However, an examination of the method used shows that the principles employed, together with the argument of de Boor [1, Corollary p.44 and Remark pp.45-46] on which the conclusion depends, can be adapted to the case of multiple knots to allow us to reach the conclusion of Lemma B. □

The proof of Lemma C requires first two supplementary results:

LEMMA 1. Let $m \in \mathbb{Z}_{++}$ and let x satisfy (2.1) and (2.4). Then $\exists\, c = c(m,x)$ such that

(5.1) $$\|e^{(k)}\|_{\ell^m_{2,x}} \le c\,(x_{k+m}-x_k)^{-m+\frac{1}{2}+k-k^-} \quad (\forall\, k \in \mathbb{Z}).$$

Proof of Lemma 1. We choose a sequence of functions $f_k(\cdot)$ such that $f_k^{(i-i^-)}(x_i) = e_i^{(k)}$, so that (see the line following (3.1) above)

$$z_i^{(k)} := e^{(k)}[x_i,\ldots,x_{i+m}] = f_k[x_i,\ldots,x_{i+m}]$$

and hence, by definition (3.4),

(5.2) $$\|e^{(k)}\|^2_{\ell^m_{2,x}} = \frac{1}{m}\sum_{i=k-m}^{k}(x_{i+m}-x_i)\,|z_i^{(k)}|^2.$$

We then estimate $z_i^{(k)} = f_k[x_i,\ldots,x_{i+m}]$ from its divided difference expansion in (3.1) to give (making use of the local mesh ratio property (2.4))

$$|z_i^{(k)}| \le c_m\,(x_{k+m}-x_k)^{-m+k-k^-} \quad (k-m \le i \le k).$$

Substituting this estimate in (5.2), and using (2.4) to estimate $(x_{i+m}-x_i)$ in terms of $(x_{k+m}-x_k)$, we then obtain (5.1). □

LEMMA 2. Let $m \in \mathbb{Z}_{++}$ and let x satisfy (2.1). Then there are constants $c > 0$, $\lambda > 0$ (which may depend on m and x) such that

$$\|L_k^{(m)}\|_{\mathcal{L}_2(-\infty,x_j]} + \|L_k^{(m)}\|_{\mathcal{L}_2[x_j,+\infty)} \le c\,\|L_k^{(m)}\|_{\mathcal{L}_2(\mathbb{R})}\cdot e^{-\lambda|k-j|} \quad (\forall\, j,k \in \mathbb{Z}).$$

Proof of Lemma 2. For simple knots, this result is due to de Boor [1, Corollary to Lemma 2, p.39]. However, by making use of a bijection map between ℓ_2 and $\mathcal{S}_{m,x} \cap \mathcal{L}_2(\mathbb{R})$ given by Jakimovski-Stieglitz [4, Theorem 3], we can follow through the pattern of de Boor's proof, carefully modified for multiple knots, to establish that his Corollary continues to hold. □

Proof of Lemma C. The object of this lemma is to determine the behaviour of $L_k(t)$ and its derivatives, for large $|t|$. In the case where all the knots are simple, de Boor [1, p.50] made use of the identity

$$L_k(t) = (t-x_{j+1})\ldots(t-x_{j+m})\,L_k[t,x_{j+1},\ldots,x_{j+m}] \quad (t < x_{j+1},\ j \ge k)$$

and then estimated $L_k(t)$ by estimating $|L_k[t,x_{j+1},\ldots,x_{j+m}]|$ in terms of $\|L_k^{(m)}\|_{\mathcal{L}_2}$. In our case we need also to estimate the intermediate derivatives which appear in the definition (3.5) of the fundamental splines, namely $L_k^{(q)}(t)$ $(0 \le q < r \le m)$. To this end we use the expression (3.1) for divided differences with multiple knots and isolate the derivative of highest order in terms of the rest of the expression to give, when x satisfies (2.4),

$$(x_{j+m}-x_j)^q|L_k^{(q)}(t)| \le c\,(x_{j+m}-x_j)^m\,|L_k[\overbrace{t,\dots,t}^{q+1},x_{j^{++}+q+1},\dots,x_{j^{++}+m}]| + \tag{5.3}$$
$$+\ c\sum_{\nu=0}^{q-1}(x_{j+m}-x_j)^\nu|L_k^{(\nu)}(t)|$$

for $j \ge k$, $0 \le q < m$, $x_{j^+} \le t \le x_{j^++1}$, $j^{++} := (j^++1)^+$.

When $q = 0$, the summation on the right of (5.3) is to be omitted.

Now by using the Peano formula (3.3) for $L_k[t,\dots,t,x_{j^{++}+q+1},\dots,x_{j^{++}+m}]$, applying the Buniakowski-Schwarz inequality, and estimating the norm of the B-spline by [4, Theorem 3(i)(p=2)] , we arrive at

$$|L_k[t,\dots,t,x_{j^{++}+q+1},\dots,x_{j^{++}+m}]| \le c(x_{j+m}-x_j)^{-\frac{1}{2}}\|L_k^{(m)}\|_{\mathfrak{L}_2[x_j,+\infty)}$$
$$\le c_1(x_{j+m}-x_j)^{-\frac{1}{2}}\|L_k^{(m)}\|_{\mathfrak{L}_2(\mathbb{R})}\cdot e^{-\lambda|k-j|} \quad \text{by Lemma 2.} \tag{5.4}$$

Next we apply (3.6) and (5.1) to (5.4), and feed the result back into (5.3) to get

$$(x_{j+m}-x_j)^q|L_k^{(q)}(t)| \le c\left(\frac{x_{j+m}-x_j}{x_{k+m}-x_k}\right)^{m-\frac{1}{2}}(x_{k+m}-x_k)^{k-k^-}\,e^{-\lambda|k-j|}\ + \tag{5.5}$$
$$+\ c\sum_{\nu=0}^{q-1}(x_{j+m}-x_j)^\nu|L_k^{(\nu)}(t)|\ .$$

Finally (for $j \ge k$) we apply the full force of our supposition (2.3) (namely that the mesh ratio should have smaller than exponential growth — we use (2.5)) together with induction on q (the case $q = 0$ comes directly from (5.5) because there is then no summation term) to deduce from (5.5) that

$$(x_{j+m}-x_j)^q\ |L_k^{(q)}(t)| \le c\,(x_{k+m}-x_k)^{k-k^-}\,e^{-\lambda_1|k-j|} \tag{5.6}$$

for $j \ge k$, $0 \le q < m$, $x_{j^+} \le t \le x_{j^++1}$, where (by the supposition (2.3)) we may choose ε sufficiently small that $\lambda_1 := \lambda - (m-\frac{1}{2})\varepsilon > 0$.

A similar procedure can now be initiated for the case $j < k$, whence we arrive at the same result (5.6), and Lemma C therefore follows. □

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