КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 439–444

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.439-444

ASYMPTOTIC BEHAVIOUR OF SPECTRAL FUNCTION FOR HARMONIC OSCILLATOR

Georgi E. Karadzhov

1. Introduction. The Harmonic Oscillator $P=-\Delta+|x|_a^2$, where $|x|_a^2=\sum_{j=1}^n a_j^2x_j^2$, $a_j\geq 0$, is essentially self-adjoint operator in $L^2\mathbb{R}^n$. Therefore $P=\int\lambda dE_\lambda$. By definition, the kernel of E_λ is the spectral function, $e(\lambda,x,y)$, for P. The asymptotic of this function is well known in the case when x and y vary on some compact [1]. Namely,

$$e(\lambda,x,y)=(2\pi)^{-n}\int_{z^2<\lambda}\exp i(x-y)z\,dz+O(\lambda^{(n-1)/2}). \tag{1}$$

Here the main term corresponds to the principal symbol z^2 of the operator P, considered locally with respect to the variable x. If x varies globally on the whole space $\mathbb{R}^n$, then the principal symbol of P is $z^2+|x|_a^2$.

The main goal of the paper is to prove an analog of (1) in the case when x and y vary globally. In addition, only the case x=y will be considered here.

2. Results.

Theorem 1. If $|x|_a<1$, then for every $N>n/2+1$ and $\lambda>1$

$$e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)=a_n(x)\lambda^{n/2}+b_n(x)\,O(\lambda^{n/2-1})+R_N(\lambda,x), \tag{2}$$

where $a_n(x)=(2\pi)^{-n}\mathrm{vol}\{x\in\mathbb{R}^n: |x|<1\}(1-|x|_a^2)^{n/2}$, $b_n(x)=(1-|x|_a^2)^{(n-3)/2}$ and $R_N(\lambda,x)=(1-|x|_a^2)^{n/2+1-2N}\,O(\lambda^{n/2+1-N})$.

Theorem 2. If $|x|_a=1$ and n is odd, then for $\lambda>1$

$$e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)=a_n(x)\lambda^{n/6}+O(\lambda^{n/6-1/3}), \tag{3}$$

where $a_n(x)=b_n(\sum_{j=1}^n a_j^4 x_j^2)^{n/6}$, b_n=const.

Theorem 3. If $|x|_a>1$, then for every $N>n/2+1$ and $\lambda>1$

$$(4)\qquad e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)=(|x|_a^2-1)^{n/2+1-2N}\,O(\lambda^{n/2+1-N})\quad\text{if } |x|_a\le 2$$

and

$$(5)\qquad e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)=(|x|_a^2)^{n/2+1-N}\,O(\lambda^{n/2+1-N})\quad\text{if } |x|_a\ge 2.$$

Theorem 4. Let $c=1/\mathrm{ch}\, a$, where $a=\max a_j$ and $\lambda>1$. Then

$$(6)\qquad e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)\le e(4\pi)^{-n/2}\exp(-c|x|_a^2)\,\lambda^{n/2}.$$

The proof is based on usual ideas applied to investigations of asymptotics of the spectral function of an elliptic operator [1]. The main tool are the following results.

Tauberian theorem. Let the function $E(\lambda,s)$, $\lambda\in\mathbb{R}$, with parameter s, be such that $|E(\lambda,s)|\le \mathrm{const}\,|\lambda|^m$ for some $m>0$ and

$$(i)\qquad \int r(\lambda-\mu)E(\mu,s)\,d\mu=a(\lambda,s)+\sum_{k=1}^p b_k(s)\,O(\lambda^{c_k}),\quad \lambda>1,$$

where c_k are real numbers, $b_k>0$, and $r\in S(\mathbb{R})$ is arbitrary. If

$$(ii)\qquad |E(\lambda+\sigma,s)-E(\lambda,s)|=\sum_{k=1}^p b_k(s)\,O(\lambda^{c_k}),\quad |\sigma|\le 1,\ \lambda>1,$$

then

$$(iii)\qquad E(\lambda,s)=a(\lambda,s)+\sum_{k=1}^p b_k(s)\,O(\lambda^{c_k}),\quad \lambda>1.$$

Morse's lemma with global parameter. Let $f(z,s)$ be a smooth function on $U\times V$, where U is an open neighborhood of zero in $\mathbb{R}^k$ and V is an open bounded domain in $\mathbb{R}^m$. Let $d(s)$ be a continuous function on the compact $\overline{V}$ such that $d(s)>0$ if $s\in V$. Assume that 0 is a critical point for the function $z\to f(z,s)$ and let $|\mu_j(s)|\ge$ const $d(s)$, $1\le j\le k$, where $\mu_j(s)$ are eigenvalues of $\partial_z^2 f(0,s)$. Then there exists a smooth change of variables $T: w\to z(w,s)$, defined on a compact of the form $Q_a=\{w\in\mathbb{R}^k : |w_j|\le a\, d(s),\ 1\le j\le k\}$ so that

$f(z(w,s),s)=f(0,s)+\frac{1}{2}\sum_{j=1}^k \mu_j(s)\, w_j^2$, $\det T'(0)=1$, $|\partial_w^p z(w,s)|\le$ const$[d(s)]^{1-|p|}$, and $Q_b\subset T\,Q_a\subset U$ for some $b>0$.

Proof of theorem 1. To prove (i) and (ii) for the function $E(\lambda,x)=e(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)$ we use the method of Hörmander [1], [2]. The Cauchy problem $(\partial/\partial t + iP)U=0$, $U(0)=I$, has a solution $U(t)$ with kernel $U(t,x,y)$, which can be written explicitly

$$U(t,x,y)=\int \exp\left[-i\sum_{j=1}^{n}(z_j^2(2a_j)^{-1}\sin 2a_jt + \tfrac{1}{2}a_j\operatorname{tg} a_jt)+i(x-y)z\right] Dz ,$$

where $Dz=(2\pi)^{-n}dz$. Let $e'_r(\lambda,x,y)=\int r(\lambda-\mu)\, de(\mu,x,y)$, $e_r(\lambda,x,y)=\int r(\lambda-\mu)e(\mu,x,y)\, d\mu$, where $r\in S(\mathbb{R})$. From the identity $e'_r(\lambda,x,y)=(2\pi)^{-1}\int \hat{r}(t)U(t,x,y)\exp i\lambda t\, dt$, where $\hat{r}(t)=\int r(s)\exp(-ist)\, ds$ and $\operatorname{supp}\hat{r}(t)\subset(-T,T)$, T is small, and integrating by parts, we get

(7) $$e_r(\lambda,\sqrt{\lambda}x,\sqrt{\lambda}x)=\lambda^{n/2+1}\int p(t,z)\exp i\lambda f(t,z,x)\, dtdz,$$

where $f(t,z,x)=t-\sum_{j=1}^{n}(z_j^2(2a_j)^{-1}\sin 2a_jt + a_jx_j^2\operatorname{tg} a_jt)$ and $p(t,z)=(2\pi)^{-1}\frac{1}{n}\, r(t)\sum_{j=1}^{n}z_j^2(a_jt)^{-1}\sin 2a_jt\,.(2\pi)^{-n}$.

To evaluate the integral (7) we shall apply the method of stationary phase with global parameter x. The critical points for the phase f are: the manifold $S(x)=\{(t,z) : t=0,\ z=d(x)\}$, where $d(x)=(1-|x|_a^2)^{1/2}$ and the points $(t(x),0)$, where $\sum a_j^2x_j^2\cos^{-2}a_jt(x)=1$. Let $\mu(x)=\mu_j(x)$ be the nonzero eigenvalues of the matrix $\partial_z^2 f$, calculated in a critical point. Then $|\mu(x)|\geq \text{const}\, d(x)$. Consequently, if $|x|_a\leq 1/2$, then the critical points are nondegenerate uniformly with respect to x. Namely, on the one hand, $S(x)$ is a nondegenerate critical manifold [3], and, on the other hand, the critical points $(t(x),0)$ are Morse's. In this case theorem 1 follows by the standard method of stationary phase [1], [3].

Let us set: $V=\{x'=(x_i,x_j,\dots,x_k) : 1/2<a_i^2x_i^2+a_j^2x_j^2+\dots+a_k^2x_k^2<1\}$, where $a_i,a_j,\dots,a_k\neq 0$ and $a_m=0$ for $m=i,j,\dots k$. Note that $\mu_j(x)$ and $d(x)$ depend only on x' and

(8) $$c\, d(x)\leq|\mu_j(x)|\leq c'\, d(x)\ ,\quad x'\in V\ ,\quad c>0\ .$$

A. Contribution from the critical points $(t(x),0)$. Let J be an integral of the form (7), where p is changed by pq, and q is a cutoff function for the domain $|z|<a\, d(x)$, $0<a<1$. In this case we have (8) for $1\leq j\leq n+1$. Therefore we can apply Morse's lemma with global parameter $x'\in V$. Consequently, the asymptotic of J is reduced to the asymptotic of the integrals K and L, where $K=\int_{Q_a} g(w,x) \exp i\frac{\lambda}{2}\sum_{j=1}^{n+1}\mu_j(x)w_j^2\, dw$ and L is an integral of the form (7), where the integration is outside the set $UQ_b(t(x),o)$ the union being taken with respect to all crutical points $(t(x),o)$.

Further, from Erdelyi's lemma [4], it follows that

(9) $$\int g(u,v)\exp i\frac{\lambda}{2}(\mu u^2+v^2\nu)\, dudv = a\,\lambda^{-1} + b\, O(\lambda^{-2}),\ \lambda>o,$$

(10) $$b=|\mu\nu|^{-1/2}(|\mu|^{-1}+|\nu|^{-1})(\|g\|_2+d\|g\|_3) + (|\mu|^{-1}+|\nu|^{-1})^2 d^2\|g\|_4,$$

(11) $$\|g\|_k=\sup|\partial^p_{u,v}g(u,v)|,\ |u|\le d,\ |v|\le d,\ |p|=k.$$

Now, applying (9) - (11) and Morse's lemma to K, we find

(12) $$K = [d(x)]^{n-3}\, O(\lambda^{-2}),\ \lambda>o.$$

To estimate L we shall integrate by parts with the help of the operator Q, where ${}^tQ=(\partial_t f)^{-1}\partial_t$. Note that $|\partial_t f(t,z,x)|\ge c[d(x)]^2$ if $(t,z)\in \operatorname{supp} p$, and $|z|\le a\, d(x)$, a is sufficiently small. Moreover, we have

(13) $$Q^k g = \sum_{j=o}^{k}A_j^k(\partial_t f)^{-2k+j}\,\partial_t^j g,\quad |A_j^k|\le C(1+z^2)^{k-j}.$$

Thus we obtain

(14) $$L = [d(x)]^{n+2-4N}\, O(\lambda^{-N}),\quad N>0.$$

Finally, from (12) and (14) it follows that

(15) $$J = [d(x)]^{n-3}\, O(\lambda^{-2}) + [d(x)]^{n+2-4N}\, O(\lambda^{-N}),\quad N>0.$$

B. Contribution from the critical points $S(x)$. Let J' be an integral of the form (7), where p is changed by $p(1-q)$. To evaluate J' it is sufficient to consider the integral

(16) $$J = \frac{1}{2}\int p(t,rs)(1-q)(rs)\, r^{n-1}\exp i\lambda f(t,rs,x)\, dtdr,$$

where $z=rs$, $r=|z|$ are polar coordinates. The critical points for the phase $(t,r)\to f(t,rs,x)$, lying on the $\operatorname{supp} p(1-q)$, coincide with $t=o$, $r=d(x)$. Moreover, if $\mu(x)$ and $\nu(x)$ are eigenvalues of the matrix $\partial^2_{t,r}f$ in the critical point, then $\mu(x)=2d(x)$, $\nu(x)=-2d(x)$. Consequently, we can apply Morse's lemma with parameter $x'\in V$. We have

(17) $$J = K + L,$$

where $K=\int_{Q_a} g(u,v,x)\exp i\frac{\lambda}{2}(\mu u^2+\nu v^2)\, dudv$, $g(o,o,x)=\frac{2}{n}(2\pi)^{-n-1}[d(x)]^{n+1}$ and L is an integral of the form (16), the integration being taken outside Q_a.

Now, using (9) - (11) and Morse's lemma, we get

(18) $$K = (2\pi)^{-n} n^{-1} [d(x)]^n \lambda^{-1} + [d(x)]^{n-3} O(\lambda^{-2}), \quad \lambda > o .$$

To estimate L we represent it as $L_1 + L_2$, where

$$L_j = \int h_j \exp i\lambda f \, dtdr \quad \text{and} \quad \operatorname{supp} h_1 \subset \{(t,r) : |r| \leq m d(x), |t| \leq T\} \setminus Q_b .$$

We integrate by parts by the operator S, where

$$^tS = H^{-1}(\partial_t f \partial_t + \partial_r f \partial_r), \quad H = (\partial_t f)^2 + (\partial_r f)^2 .$$

Note that $H(t,r,x) \geq \text{const} [d(x)]^4$ if $(t,r) \in \operatorname{supp} h_1$, $x' \in V$ and

$S^k g = \sum_{j=o}^{k} A_j^k H^{-k+j/2} (\partial_t + \partial_r)^j g$, $|A_j^k| \leq \text{const}$. From here we obtain

(19) $$L_1 = [d(x)]^{n+2-4N} O(\lambda^{-N}), \quad N > 0 .$$

To estimate L_2 we note that for $m \gg$ $|\partial_t f| \geq \text{const} [d^2 + r^2]$ if $(t,r) \in \operatorname{supp} h_2$, $x' \in V$. It follows that we can integrate by parts with the help of the operator Q, using the identity (13). Thus

(20) $$L_2 = [d(x)]^{n+2-4N} O(\lambda^{-N}), \quad N > n/2+1 .$$

Finally, from (19) and (20) follows

(21) $$L(\lambda,x,s) = [d(x)]^{n+2-4N} O(\lambda^{-N}), \quad N > n/2+1 ,$$

and from (7), (15)-(18) and (21) we obtain

(22) $$e_r(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}x) = a_n(x)\lambda^{n/2} + b_n(x) O(\lambda^{n/2-1}) + R_N(\lambda,x), \quad N > n/2+1$$

Analogously,

(23) $$e_r'(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}x) = b_n(x) \, O(\lambda^{n/2-1}) + R_N(\lambda,x) .$$

Now theorem 1 follows from (22),(23) by the tauberian theorem and taking into account the estimate (6).

4. Sketch of proof of theorem 2. If $|x|_a = 1$, then only the point $t=o$, $z=o$ is a critical for the phase f. By smooth change of variables f can be reduced to the form $u(u^2+v^2)$. Since n is odd, in polar coordinates $v = rs$, $r = |v|$, we have to evaluate an integral of the form

(24) $$J = \int g(u,r,x) \, r^{n+1} \exp i\lambda u(u^2+r^2) \, dudr ,$$

where $g(o,o,x) = \frac{1}{n}(2\pi)^{-n-1} (\frac{1}{3} \sum_{j=1}^{n} a_j^4 x_j^2)^{n/6}$. By the method of stationary phase from (24) follows (3) and

$$b_n = (2\pi)^{-n-1}(n+1)!\ 3^{n/3-1}\ 2^{-2n/3-1}\ \mathrm{vol}\left\{x\in\mathbb{R}^n : |x|<1\right\} c_n ,$$

$$c_n = \sum_{k+j=n+1}(-1)^j \frac{1}{k!}\frac{1}{j!}\Gamma\left(\frac{k+1}{3}\right)\Gamma\left(\frac{j+1}{3}\right)\left[\exp i\pi\frac{k+1}{6} + (-1)^k \exp\left(-i\pi\frac{k+1}{6}\right)\right] .$$

5. Sketch of proof of theorem 3. We integrate by parts in (7) by the operator Q, taking into account (13) and the estimates: $|\partial_t f| \geq c(d^2+z^2)$ if $|x|_a \leq 2$ and $|\partial_t f| \geq c(|x|_a^2+z^2) \geq |A_j^k|$ if $|x|_a \geq 2$.

6. Proof of theorem 4. The function $V(t,x,y)=\int_0^\infty e^{-\lambda t}\, de(\lambda,x,y)$ is a solution of Cauchy's problem $(\partial/\partial t + P)\, V(t,x,y) = 0$, $V(0,x,y) = \delta(x-y)$, and moreover, we can write it explicitly:

$$V(t,x,y)=\prod (a_j^{-1}2\pi \mathrm{sh}2a_j t)^{-1/2}\exp\left[-a_j\left(\tfrac{1}{2}(x_j^2+y_j^2)\mathrm{ch}2a_j t - x_j y_j\right)\mathrm{sh}^{-1}2a_j t\right].$$

Therefore $\int_0^\infty e^{-\lambda t}\, de(\lambda,x,x) \leq (4\pi t)^{-n/2}\exp(-c|x|_a^2 t)$ if $0<t<1$. From here we get (6) .

References

1. L. Hörmander. The spectral function of an elliptic operator. Acta Math. 121, 1968, 193 - 218 .

2. B. Helffer and D. Robert. Comportement asymptotique precise du spectre d'opérateurs globalement elliptiques dans R^n. Sem. Goulaouic - Meyer - Schwartz, 1980 - 1981, exposé no 11, 1 - 22 .

3. J. Chazarain. Formule de Poisson pour les variétés riemanniennes. Invent. Math. 24, 1974, 65 - 82 .

4. M. Fedoryuk. Method perevala. Moskva, 1977 (in Russian) .

Institute of Mathematics with Computer Center
Bulgarian Academy of Sciences
P. O. Box 373 1090 Sofia Bulgaria