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ASYMPTOTICS OF ERROR FOR INTERPOLATING SPLINES OF EVEN DEGREE

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1. Introduction. In this paper we present an extension of the results on asymptotic behaviour of the splines of odd degree which were obtained by the author [1-3], to the case of periodic splines of even degree.

Let in the knots of a mesh $\Delta: x_i = a + ih$, $i=0,\dots,N$, with $h=(b-a)/N$ be given the values $f_i = f(x_i)$ of some $(b-a)$-periodic function $f(x)$. Denote by $\bar{\Delta}$ a mesh $\bar{\Delta}: \bar{x}_i = (x_{i-1}+x_i)/2$, $i=1,\dots,N$.

We define the interpolating spline $S(x)$ of degree $2r$ $(r=1,2,\dots)$ of deficiency 1 as a function which agrees with some polynomials of degree not greater than $2r$ on each interval $[a,\bar{x}_1]$, $[\bar{x}_i,\bar{x}_{i+1}]$, $i=1,\dots,N-1$, $[\bar{x}_N, b]$, and interpolates the values f_i on the mesh Δ, i.e. $S(x_i)=f_i$, $i=0,\dots,N$, and belongs to the class $C^{2r-1}[a,b]$. Suppose that $S(x)$ satisfies periodic boundary conditions: $S^{(k)}(a)=S^{(k)}(b)$, $k=1,\dots,2r$. The existence and uniqueness of such a spline were proved in [4].

In the present paper we obtain the asymptotic expansion in powers of h of the difference $S^{(p)}(x)-f^{(p)}(x)$, $p=0,\dots,2r$. Besides, we give the asymptotic formula for the jump of the highest spline derivative. Note that similar results are found in [5] for the special case of a parabolic spline $(r=1)$. On the basis of asymptotics we construct a parametric family of splines which secure the higher order of approximation than the original interpolating spline.

2. Main results. We denote $S_i^{(k)} = S^{(k)}(x_i)$, $f_i^{(k)} = f^{(k)}(x_i)$, $\|f\| = \sup_{a\le x\le b}|f(x)|$

and let $\tilde{C}^k[a,b]$ be a class of $(b-a)$-periodic functions $f(x)\in C^k(-\infty,\infty)$

Lemma. Let $f\in \tilde{C}^{2r+3}[a,b]$. Then

$$S_i^{(p)} = f_i^{(p)} + \frac{(-1)^p B_{2q}\left(\frac{1}{2}\right)}{2q\,(2r-p)!} h^{2q} f_i^{(2q+p)} + \alpha_{i,p} \qquad (1)$$
$$(p=1,\dots,2r),$$

where $\max_i |\alpha_{i,p}| = O(h^{2r+3-p})$, $q=[\,(2r+2-p)/2\,]$ ($[u]$ is the integer part of u), $B_m(t)$ is the m-th Bernoulli polynomial [6].

Lemma is proved by the same method as [3, corollary of Theorem 1] We need the boundedness of $\|A^{-1}\|_\infty$ by a constant independent of N, where A is a matrix of the system from which the values $S_i^{(p)}$ are determined. The first defining row of A (A is a circulant matrix of order N) is

$$(C(2r,0,r), C(2r,0,r+1),\dots,C(2r,0,2r),0,\dots,0,C(2r,0,0),\dots,C(2r,0,r-1))$$

Here $C(2r,0,\cdot)$ are the coefficients in linear relations, which connect the values S_i and $S_i^{(p)}$ [7]. This result follows from the estimate $\|A^{-1}\|_\infty \le 1/|E_{2r}|$ (E_k is the k-th Euler number [6]), which is established by analogy with [8] by using [9, Lemma 5]

We add to the mesh $\bar{\Delta}$ knots $\bar{x}_0 = a-h/2$, $\bar{x}_{N+1} = b+h/2$.

Theorem 1. Let $f\in\tilde{C}^{2r+3}[a,b]$. Then for all $x\in[\bar{x}_i,\bar{x}_{i+1}]\cap[a,b]$, $i=0,\dots,N$, we have

$$S^{(p)}(x) - f^{(p)}(x) = -\frac{B_{2r+1-p}(t)}{(2r+1-p)!} h^{2r+1-p} f^{(2r+1)}(x) + $$
$$+ D_p(t) h^{2r+2-p} f^{(2r+2)}(x) + O(h^{2r+3-p}),\quad p=0,\dots,2r \qquad (2)$$

uniformly with respect to $x\in[a,b]$, where $t=(x-\bar{x}_i)/h$,

$$D_0(t) = \frac{(2r+1)\left(B_{2r+2}(t) - B_{2r+2}\left(\frac{1}{2}\right)\right)}{(2r+2)!},\quad D_p(t) = \frac{(2r+1-p)\,B_{2r+2-p}(t)}{(2r+2-p)!}\quad (p\ge 1)$$

Proof. Let $x\in[\bar{x}_i,\bar{x}_{i+1}]$. Using in $E(x)=S(x)-f(x)$ the Taylor expansion about x_i we obtain

$$E(x)=S(\bar{x}_i+th)-f(\bar{x}_i+th)=\sum_{k=0}^{2r}\frac{((t-1/2)h)^k}{k!}S_i^{(k)}-$$

$$-\sum_{k=0}^{2r+2}\frac{((t-1/2)h)^k}{k!}f_i^{(k)}-\frac{1}{(2r+2)!}\int_{x_i}^{x}f^{(2r+3)}(v)(x-v)^{2r+2}dv$$

Substituting in the right-hand side of this equality expression (1) for $S_i^{(p)}$ and combining the terms with the same power of h, we find

$$E(x)=-\left\{\frac{(t-1/2)B_{2r}(\frac{1}{2})}{1!\,(2r)!}+\frac{(t-1/2)^3B_{2r-2}(\frac{1}{2})}{3!\,(2r-2)!}+\dots+\right.$$

$$\left.+\frac{(t-1/2)^{2r-1}B_2(\frac{1}{2})}{(2r-1)!\,2!}+\frac{(t-1/2)^{2r+1}}{(2r+1)!}\right\}h^{2r+1}f_i^{(2r+1)}+$$

$$+\left\{\frac{(t-1/2)^2(2r-1)B_{2r}(\frac{1}{2})}{2!\,(2r)!}+\frac{(t-1/2)^4(2r-3)B_{2r-2}(\frac{1}{2})}{4!\,(2r-2)!}+\dots+\right.$$

$$\left.+\frac{(t-1/2)^{2r}B_2(\frac{1}{2})}{(2r)!\,2!}-\frac{(t-1/2)^{2r+2}}{(2r+2)!}\right\}h^{2r+2}f_i^{(2r+2)}+R(x)$$

where $R(x)=\sum_{k=1}^{2r}\frac{((t-1/2)h)^k}{k!}\alpha_{i,k}-\frac{1}{(2r+2)!}\int_{x_i}^{x}f^{(2r+3)}(v)(x-v)^{2r+2}dv$

We note that the coefficients at $h^{2r+1}f_i^{(2r+1)}$, $h^{2r+2}f_i^{(2r+2)}$ coincide with the Taylor series expansion about $t=1/2$ of the polynomials

$$-\frac{B_{2r+1}(t)}{(2r+1)!},\quad \frac{(2r+1)(B_{2r+2}(t)-B_{2r+2}(\frac{1}{2}))}{(2r+2)!}-\frac{B_{2r+1}(t)}{(2r+1)!}(t-1/2),$$

respectively. Therefore,

$$E(x)=-\frac{B_{2r+1}(t)}{(2r+1)!}h^{2r+1}f_i^{(2r+1)}+\left\{\frac{(2r+1)(B_{2r+2}(t)-B_{2r+2}(\frac{1}{2}))}{(2r+2)!}-\right.$$

$$\left.-\frac{B_{2r+1}(t)}{(2r+1)!}(t-1/2)\right\}h^{2r+2}f_i^{(2r+2)}+R(x)\qquad(3)$$

Taking into account that $B_n'(t)=nB_{n-1}(t)$ and differentiating (3) $2r$ times with respect to x, we obtain the expressions for $E^{(p)}(x)$, $p=1,\dots,2r$. Now, using the Taylor expansion about x for values $f_i^{(2r+1)}$, $f_i^{(2r+2)}$ in $E^{(p)}(x)$, $p=0,\dots,2r$ and estimating the integral terms we establish the assertion of the theorem.

Remark. From the proof it is clear that the number of the terms in (2) can be made more numerous. To do this, it is sufficient to derive the corresponding number of the terms in (1).

The form of the leading term in (2) gives the possibility to define the points of superconvergence in which the approximation order is increased. These points are connected with zeros of Bernoulli polynomials $B_{2r+1-p}(t)$, $t\in[0,1]$ (See [10] on the zeros of the Bernoulli polynomials).

The following two statements are derived directly from Theorem 1.

Theorem 2. Let $f\in \tilde{C}^{2r+2}[a,b]$. Then

$$\| S^{(p)}-f^{(p)}\| \le h^{2r+1-p}K_p\,\| f^{(2r+1)}\| + O(h^{2r+2-p})$$
$$(p=0,\dots,2r),$$

where

$$K_p=\left\{\begin{array}{ll}\dfrac{|B_{2r+1-p}|}{(2r+1-p)!}, & \text{p - odd}\\[2ex] \dfrac{\max\limits_{0\le t\le 1}|B_{2r+1-p}(t)|}{(2r+1-p)!} & \text{otherwise}\end{array}\right\} < \frac{2}{(2\pi)^{2r+1-p}(1-2^{p-2r})}$$

$B_m=B_m(0)$ is the m-th Bernoulli number [6].

Theorem 3. Under the assumptions of Theorem 1, we have

$$\max_i | S^{(p)}(\bar{x}_i+t^*h)-f^{(p)}(\bar{x}_i+t^*h)| =$$
$$=|D_p(t^*)|\,h^{2r+2-p}\max_i| f^{(2r+2)}(\bar{x}_i+t^*h)| + O(h^{2r+3-p}),\ p=0,\dots,2r,$$

where for $p\ge 1$ t^* traces the set of the zeros of $B_{2r+1-p}(t)$, $t\in[0,1]$ and $t^*=0,1$ for $p=0$.

The values of the constants K_p in Theorem 2 and their bounds follow from the bounds of the Bernoulli numbers and the Bernoulli polynomials (see [6], [10]). Similarly we may also estimate the constants $|D_p(t^*)|$. For $p=2k$, $k=0,1,\dots,r$, the constants $D_p(t^*)$ have the form

$$D_p\left(\tfrac{1}{2}\right)=\frac{(2r+1-p)(2^{p-2r-1}-1)\,B_{2r+2-p}}{(2r+2-p)!},\quad p=2,\dots,2r;$$

$$D_p(0)=D_p(1)=\begin{cases}\dfrac{(2r+1-p)B_{2r+2-p}}{(2r+2-p)!}, & p=2,\dots,2r-2,\\ (2r+1)B_{2r+2}\left[2-2^{-2r-1}\right]/(2r+2)!, & p=0.\end{cases}$$

Theorem 4. Let $f\in C^{4r+5}[a,b]$. Then

$$(S^{(2r)}(\bar{x}_i+0)-S^{(2r)}(\bar{x}_i-0))/h-f^{(2r+1)}(\bar{x}_i)=$$
$$=\frac{(2^{2r+2}-2)(2r+1)B_{2r+2}}{(2r+2)!\,2^{2r+2}}h^{2r+2}f^{(4r+3)}(\bar{x}_i)+O(h^{2r+4}).$$

The proof of this theorem is similar to [2, Theorem 2]. We need only the system of equations for the jumps of the highest spline derivative which can be easily obtained using [11].

Note that the convergence of $(S^{(2r)}(\bar{x}_i+0)-S^{(2r)}(\bar{x}_i-0))/h$ to $f^{(2r+1)}(\bar{x}_i)$ is proved in [12, Theorem 5.1]. Theorem 4 establishes the best possible order of approximation in case of sufficiently smooth $f(x)$.

3. The increase of approximation order. We define $S_{\tau_p,p}(x)$, $p=0,\dots,2r+1$, as follows:

$$S_{\tau_p,p}(x)=S^{(p)}(x)+C_p(t)h^{2r+1-p}\delta^2 S^{(2r-1)}(\tilde{x})$$
$$x\in[\bar{x}_i,\bar{x}_{i+1}]\cap[a,b],\ i=0,\dots,N$$

where $\tilde{x}=\bar{x}_i+\tau_p(t)h$, $0\le\tau_p(t)\le1$, $t=(x-\bar{x}_i)/h$, $C_p(t)=\dfrac{B_{2r+1-p}(t)}{(2r+1-p)!}$,

$$\delta^2 S^{(2r-1)}(\tilde{x})=(S^{(2r-1)}(\tilde{x}+h)-2S^{(2r-1)}(\tilde{x})+S^{(2r-1)}(\tilde{x}-h))/h^2$$

(If $\tilde{x}$, $\tilde{x}\pm h\notin[a,b]$, we determine the values $S^{(2r-1)}(\tilde{x})$, $S^{(2r-1)}(\tilde{x}\pm h)$ by periodic extension of $S(x)$).

We hote that $S_{\tau_p,p}(x)$ are determined by the explicit formulas which include the coefficients of the original interpolating spline.

The next statement follows from Theorem 1.

Theorem 5. Let $f\in C^{2r+3}[a,b]$. Then for all $x\in[\bar{x}_i,\bar{x}_{i+1}]\cap[a,b]$, $i=0,\dots,N$ we have

$$S_{\tau_p,p}(x)-f^{(p)}(x)=(D_p(t)+C_p(t)(\tau_p(t)-t))h^{2r+2-p}f^{(2r+2)}(x)+$$
$$+O(h^{2r+3-p}),\quad p=0,\dots,2r+1$$

uniformly with respect to $x,\tilde{x}$.

For any $\tau_p(t)$ the functions $S_{\tau_p,p}(x)$ approximate $f^{(p)}(x)$ for an order higher than the initial spline. The parameter $\tau_p(t)$ may be chosen due to the accuracy, or continuity of the derived constructions.

Consider some methods of the choice of $\tau_p(t)$. Let $\tau_p(t)=\tau=const.$ In this case $S_{\tau,0}(x)$ is the interpolating spline of $2r+1$ degree and of the class $C[a,b]$ (in the knots of $\bar\Delta$ the odd degree derivatives are discontinuous). Note that $S^{(p)}_{\tau,0}(x)=S_{\tau,p}(x)$, $p=0,\ldots,2r+1$. If $\tau_p(t)=t$, then for any p $S_{t,p}(x)$ is a spline of degree $2r+2-p$ (interpolating for $p=0$) and is continuous in the knots $\bar{x}_i$ for any $p=0,\ldots,2r+1$.

The influence of the choice of $\tau_p(t)$ on the accuracy was analysed in [3], where similar approach is used for the odd degree splines.

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