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OPTIMAL QUADRATURE FORMULAE FOR FUNCTION CLASSES W^rH_ω

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1. <u>Introduction.</u> This note is devoted to the optimal quadrature formulae for approximate calculation of definite integrals according to a given information, using the function values and its derivatives at a fixed number of points from the integration interval with or without restrictions on the points, for the function classes W^rH_ω ($r = 0,2,4,6,\ldots$) and W^rH_1 ($r = 1,3,5,\ldots$).

2. <u>Stating of the Problem</u>. Let n be a positive integer ($n \in \mathbb{N}$), r - nonnegative integer ($r \in \mathbb{N}_0$), $X = (x_1, x_2, \ldots, x_n)$ - a network of points in the interval $[0,1]$:

$$(1) \qquad 0 \le x_1 \le x_2 \le \ldots \le x_n \le 1 ,$$

and $\omega(t)$, $t \in [0,\infty)$, be an arbitrary modulus of continuity. By $W^rH_\omega = W^rH_\omega[0,1]$ we denote the set of all functions $f:[0,1] \to \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), which r-th derivative ($f^{(0)} = f$) satisfies the condition : for each two points x and x' from the interval $[0,1]$ the inequality

$$|f^{(r)}(x) - f^{(r)}(x')| \le \omega(|x - x'|)$$

is satisfied.

Then, by definition $H_\omega = W^0H_\omega$, and $H_\alpha = H_\omega$, if $\omega(t) = t^\alpha$, $0 < \alpha \le 1$, $t \in [0,\infty)$.

For each $f \in W^rH_\omega$ we suppose that the data

$$(2) \qquad T_r = T_r(f,X) = (f^{(k)}(x_i))_{i=1,k=0}^{n \quad r}$$

are known.

By $\mathbb{L}(T_r)$ we denote the set of all possible methods of approximate calculation of the integral

$$I(f) = \int_0^1 f(x)\, dx$$

using to information (2).

Then

$$R_n(W^rH_\omega, Q, X) = \sup\{\, |I(f) - Q(f,X)| : f \in W^rH_\omega \}$$

is said to be the error of the method Q in the class W^rH_ω using information (2).

Let us put

$$R_n(W^rH_\omega, X) = \inf\{ R_n(W^rH_\omega, Q, X) : Q \in L(T_r)\} .$$

The quadrature method $Q^* \in L(T_r)$, for which

$$R_n(W^rH_\omega, Q^*, X) = R_n(W^rH_\omega, X),$$

is said to be the best method for recovering the integral $I(f)$ in the class W^rH_ω using information (2).

We put

$$R_n(W^rH_\omega) = \inf_X R_n(W^rH_\omega, X).$$

Let $Q^*(f,X)$ be the best method for recovering the integral $I(f)$ in the class W^rH_ω using information (2). If for the function $f^* \in W^rH_\omega$ and the network X^* from the type (1) the equality

$$| I(f^*) - Q^*(f^*,X^*) | = R_n(W^rH_\omega)$$

holds, this means that the msthod $Q^*(f,X^*)$ and the network X^* are called optimal quadrature formula and optimal network for the integral $I(f)$ in the class W^rH_ω using information (2) respectively , while f^* is said to be the extremum of $I(f)$ in W^rH_ω at T_r.

The problem for finding an optimal quadrature formula for the integral $I(f)$ in the class H_α using information of the type (2) ($r = 0$) has been solved by A.H.Turetskii in 1951 in [1] , and in the class H_ω by N.P.Korneytchuk in 1968 in [2] .

3. Main Results.

Theorem 1. Let n be a positive integer, r - nonnegative even integer, and $\omega(t)$ - an arbitrary modulus of continuity. Out of all quadrature formulae for approximate calculation of the integral $I(f)$ in classes W^rH_ω ($r = 0,2,4,\dots$)using information of the type (2), the formula

$$(3)\qquad \int_0^1 f(x)dx = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{[r/2]} \frac{f^{(2k)}(\frac{2i-1}{2n})}{(2n)^{2k}.(2k+1)!} + R_n^r(f,X^*)$$

is the optimal one, where $[\alpha]$ is the integer part of α , and

$$(4)\qquad X = \left(\frac{1}{2n}, \frac{3}{2n}, \frac{5}{2n}, \dots, \frac{2n-1}{2n}\right) .$$

Moreover

$$R_n(H_\omega) = \int_0^1 \omega\left(\frac{u}{2n}\right)du , \quad r = 0,$$

$$R_n(W^rH_\omega) = \frac{1}{(2n)^r(r-1)!}\int_0^1\int_0^1 u^r(1-z)^{r-1}\omega(uz/(2n))dzdu, \quad r=2,4,\ldots$$

The function $f^* \in W^rH_\omega$ ($r=0,2,4,\ldots$) for which

$$f^{*(r)}(x) = \omega(|x-\frac{2i-1}{2n}|), \quad x\in\left[\frac{i-1}{n}, \frac{i}{n}\right], \quad i=\overline{1,n},$$

is the extremal one.

Remark. The result of the theorem, for $r=0$ is due to N.P.Korneytchuk [2] .

Theorem 2. Let $n\in N$ and $r>0$ be an odd integer. Out of all quadrature formulae, (3) is the optimal one for the integral $I(f)$ in classes W^rH_1 ($r=1,3,5,\ldots$)using information of the type (2).

Moreover

$$R_n(W^rH_1) = ((2n)^{r+1}(r+2)!)^{-1}.$$

The function $f^*\in W^rH_1$ ($r=1,3,5,\ldots$)

(5) $$f^*(x) = x^{r+1}/(r+1)! + a_1x^r + \ldots + a_{r+1}, \quad a_i=\text{const.}, \quad (i = \overline{1,r+1})$$

is the extremal one.

Theorem 3. Let $n\in N$ and $r\geq 0$ be an even integer, and $\omega(t)$ is an arbitrary modulus of continuity. Out of all quadrature formulae for $I(f)$, using information of the type (2) with the additional restriction

(6) $$x_1 = 0$$

on the network X , optimal in the class W^rH_ω ($r=0,2,4,\ldots$) is the formula

(7) $$\int_0^1 f(x)dx = \frac{1}{2n-1}\sum_{k=0}^{r}\frac{f^{(k)}(0)}{(2n-1)^k(k+1)!} +$$

$$+ \frac{2}{2n-1}\sum_{i=2}^{n}\sum_{k=0}^{[r/2]}\frac{1}{(2n-1)^{2k}(2k+1)!}.f^{(2k)}\left(\frac{2i-2}{2n-1}\right) + R_n^r(f,X_1),$$

where

(8) $$X_1 = (0,2/(2n-1),4/(2n-1), \ldots ,(2n-2)/(2n-1)).$$

Moreover

$$R_n(H_\omega) = \int_0^1\omega\left(\frac{u}{2n-1}\right)du, \quad r = 0,$$

$$R_n(W^rH_\omega) = \frac{1}{(2n-1)^r(r-1)!}\int_0^1\int_0^1 u^r(1-z)^{r-1}\omega\left(\frac{uz}{2n-1}\right)dzdu, \quad r=2,4,\ldots$$

Theorem 4. Let $n \in \mathbb{N}$, and $r > 0$ be an odd number. Out of all quadrature formulae for the integral $I(f)$, using information of the type (2) with additional restriction (6) on the **network** X, optimal in the classes W^rH_1 ($r = 1,3,5, \ldots$) is the formula (7) with the error

$$R_n(W^rH_1) = ((2n-1)^{r+1}(r+2)!)^{-1} .$$

Theorem 5. Let $n \geq 2$ be an integer, $r \geq 0$ - an even integer, and $\omega(t)$ - an arbitrary modulus of continuity. Out of all quadrature for the integral $I(f)$, using information of type (2), at the additional restriction

(9) $$x_1 = 0 , x_n = 1 ,$$

on the network X, optimal in the class W^rH_ω ($r = 0,2,4,\ldots$) is the formula

(10) $$\int_0^1 f(x)dx = \frac{1}{2n-2} \sum_{k=0}^{r} \frac{f^{(k)}(0) + (-1)^k f^{(k)}(1)}{(2n-2)^k (k+1)!} +$$

$$+ \frac{1}{n-1} \sum_{i=2}^{n-1} \sum_{k=0}^{[r/2]} \frac{f^{(2k)}((i-1)/(n-1))}{(2n-2)^{2k}(2k+1)!} + R_n^r(f,X_2) ,$$

where

(11) $$X_2 = (0, 1/(n-1), 2/(n-1), 3/(n-1), \ldots , 1) ,$$

with an error

$$R_n(H_\omega) = \int_0^1 \omega(u/(2n-2))du \quad \text{for } r = 0,$$

or

$$R_n(W^rH_\omega) = \frac{1}{(2n-2)^r (r-1)!} \int_0^1 \int_0^1 u^r (1-z)^{r-1} \omega\left(\frac{uz}{2n-2}\right) dz\,du$$

for $r = 2,4,6,\ldots$.

Theorem 6. Let $n \geq 2$ be an integer, and $r > 0$ - an odd integer. Out of all quadrature formulae for the integal $I(f)$, using information of type (2), with the additional restriction (9) of the network X, optimal in the classes W^rH_1 ($r = 1,3,5, \ldots$) is the formula (10), with an error

$$R_n(W^rH_1) = ((2n-2)^{r+1}(r+2)!)^{-1} .$$

Remark : 1. Function (5) is extremal for theorems 4 and 6.
2. The extremal functions for theorems 3 and 5 are also known and will be published elsewhere.
3. The optimal quadrature formulae for the classes W^rH_ω for an odd $r > 0$, are not known.

4. Proof of the Main Results. It is based on the following optimisation theorem :

Theorem 7. Let $A = (a_0, a_1, \dots, a_n)$ and $X = (x_1, \dots, x_n)$ are networks of points in the interval $[0,1]$, with points satisfying condition

(12) $$0 = a_0 \le a_1 \le \dots \le a_n = 1 \ , \ 0 \le x_1 \le x_2 \le \dots \le x_n \le 1 \ ,$$

the function $\varphi : [0,1] \to \mathbb{R}$ is (strongly) increasing and let us put

$$L_n(\varphi, A, X) = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \varphi(|x - x_i|)dx \ .$$

Then : a) For each pair of networks (A,X) of the type (12) the inequality

$$L_n(\varphi, A, X) \ge \int_0^1 \varphi(u/(2n))du$$

holds. The equality can be reached (only) for pair (A^*, X^*), where X^* can be determined as in (4), and $A^* = (0, 1/n, 2/n, \dots, 1)$.

b) For each pair of networks (A,X) of the type (12) satisfying restriction (6) the inequality

$$L_n(\varphi, A, X) \ge \int_0^1 \varphi(u/(2n-1))du$$

holds. The equality can be reached (only) for pair of networks (A_1, X_1), where X_1 is determined by (8), and $A_1 = (0, 1/(2n-1), 3/(2n-1), \dots, 1)$.

c) For each pair of networks (A,X) of the type (12), satisfying restriction (9), the inequality

$$L_n(\varphi, A, X) \ge \int_0^1 \varphi(u/(2n-2))du \ , \ n > 1 \ ,$$

holds. The equality can be reached (only) for pair of networks (A_2, X_2), where X_2 can be determined as in (11), and

$$A_2 = (0, 1/(2n-2), 3/(2n-2), \dots, (2n-3)/(2n-2), 1).$$

The proof of this theorem will be given in another item.

Now we are going to give as an illustration the scheme of the proof of the theorem 3, at $r = 2,4,\dots$ (the case $r = 0$ can be seatled in a similar way).

On the given network of the type (1) we can build a new network $C=(c_0,c_1,\ldots,c_n)$ according to the formulae

$$c_0 = 0,\ c_i = (x_i + x_{i+1})/2\ ,\ i = \overline{1,n-1}\ ,\ c_n = 1.$$

Using the Taylor's formula with remainder in the form of an integral we obtain

$$I(f) = Q^r(f,X) + R^r_n(f,X)\ ,$$

where

$$(13)\qquad Q^r(f,X)=\sum_{i=1}^{n}\sum_{k=0}^{r}\frac{(c_i-x_i)^{k+1}-(c_{i-1}-x_i)^{k+1}}{(k+1)!}.f^{(k)}(x_i)\ ,$$

$$(14)\quad R^r_n(f,X)=\sum_{i=1}^{n}\int_{c_{i-1}}^{c_i}\frac{(x-x_i)^r}{(r-1)!}.\int_0^1(1-z)^{r-1}\left[f^{(r)}(x_i+(x-x_i)z)-f^{(r)}(x_i)\right]dzdu\ .$$

From (14) it follows the estimation

$$(15)\qquad \sup\left\{|R^r_n(f,X)|:\ f\in W^rH_\omega\right\}\le\frac{1}{(r-1)!}L_n(\varphi,C,X)\ ,\quad r\in\mathbb{N}\ ,$$

where

$$\varphi(u)= u^r\int_0^1(1-z)^{r-1}\omega(uz)dz\ ,\quad u\in[0,1],$$

is an increasing function.

According to the corollary of Smolyak's lemma [3](see also [4]) for the best recovering the integral $I(f)$ a method using information (2), at $r=2,4,\ldots$ can easily be obtained:

$$(16)\qquad R_n(W^rH_\omega,X) = \frac{1}{(r-1)!}L_n(\varphi,C,X)\ .$$

From (15) and (16) it follows that (13) is the best method for recovering using information (2) and the equality :

$$(17)\ \sup\left\{|R^r_n(f,X)|:\ f\in W^rH_\omega\right\}= R_n(W^rH_\omega,X) = ((r-1)!)^{-1}.L_n(\varphi,C,X)$$

holds.

By applying of theorem 7 b) to (17), we obtain the statements of theorem 3 for $r=2,4,\ldots$.

References

1. А.Х.Турецкий. Об оценках приближений квадратурными формулами для функций, удовлетворяющих условию Липшица. УМН 6, вып.5, 1951, 166-171.

2. Н.П.Корнейчук. Наилучшие кубатурные формулы для некоторых классов функций многих переменных. Матем. заметки, 3, № 5, 1968, 565-576.

3. С.А.Смоляк. Об оптимальном восстановлении функций и функционалов от них. Канд. дисс., Москва, 1965.

4. Н.С.Бахвалов. Об оптимальности ленейных методов приближения операторов на выпуклых классах функций. Ж. выч. мат. и мат. физ., 11, № 4, 1971, 1014-1018.

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