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AN APPLICATION OF MEIJER'S G-FUNCTION TO BESSEL-TYPE DIFFERENTIAL OPERATORS

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The general Bessel-type operator of order m is said to be the singular linear differential operator

/1/ $$B = t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\cdots\frac{d}{dt}t^{\alpha_m}, \quad 0 \le t \le \infty .$$

Several approaches to operational calculi have been proposed by I. Dimovski [1], [2], [3]. Such operational calculi can be used for treating of singular Cauchy problems containing Bessel-type operators /1/. Such kind of problems arise quite often in mathematical physics : in axisymmetric problems of potential theory, in elasticity theory, in hydrodynamics etc.

Our aim is to cast the earlier results on Bessel-type operational calculi in a new sight : to represent them in terms of integral operators with Meijer's G-functions as kernels.

Here for convenience we recall the definition of the generalized hypergeometric function due to C. S. Meijer as a Mellin-Barnes type integral

/2/ $$G^{m,n}_{p,q}(z) = G^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{k=1}^{m}\Gamma(b_k-s)\prod_{k=1}^{n}\Gamma(1-a_k+s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+s)\prod_{k=n+1}^{p}\Gamma(a_k+s)}\, z^s\, ds ,$$

where $o \le m \le q$, $o \le n \le p$ and L is a suitably chosen contour in $\mathbb{C}$ separating the poles of $\Gamma(b_j-s)$, $j=1,2,\ldots,m$ from the poles of $\Gamma(1-a_k+s)$, $k=1,2,\ldots,n$ in the integrand.

The role of one definite type Meijer's G-function ($q=m$, $n=o$) in the theory of the operators of the form /1/ is seen from the following

Theorem 1. The function

$$/3/ \qquad y(t) = G^{m,o}_{o,m}\left[\frac{t^{\beta}}{(-\beta)^m} \,\middle|\, (-\gamma_{m-k+1})_{k=1}^{m}\right]$$

with

$$/4/ \qquad \gamma_k = \frac{1}{\beta}(\alpha_k + \ldots + \alpha_m - m + k)\ , \quad k=1,2,\ldots,m$$

is a solution of the linear differential equation

$$/5/ \qquad B\, y(\lambda t) = \lambda^{\beta} y(\lambda t)\ , \quad \lambda \text{ being a real parameter.}$$

Proof. Let us consider the linear differential equation of the form

$$/6/ \qquad \left[(-1)^{p-m-n}\, x \prod_{k=1}^{p} (\delta_x - a_k + 1) - \prod_{k=1}^{q} (\delta_x - b_k)\right] u(x) = 0\ ,$$

where $\delta_x = x\frac{d}{dx}$,its order being equal to max (p,q) . The function $G^{m,n}_{p,q}\left[x \,\middle|\, {a_k \atop b_k}\right]$ satisfies the equation /6/ (see [4],p.210). Therefore the function

$$u(x) = G^{m,o}_{o,m}\left[x \,\middle|\, b_k\right]$$

is a solution of the equation

$$/7/ \qquad \left[(-1)^m x - \prod_{k=1}^{m} (\delta_x - b_k)\right] u(x) = 0\ .$$

But after the substitution $x = \frac{\lambda^{\beta} t^{\beta}}{(-\beta)^m}$ and simple calculations , /7/ gets the form

$$\left[\lambda^{\beta} - t^{-\beta} \prod_{k=1}^{m} (\delta_t - \mu_k)\right] y(\lambda t) = 0\ ,$$

or

$$/7'/ \qquad \left[t^{-\beta} \prod_{k=1}^{m} (\delta_t - \mu_k)\right] y(\lambda t) = \lambda^{\beta} y(\lambda t)\ ,$$

where $\delta_t = t\frac{d}{dt}$ and $\mu_k = \beta b_k$. Let us choose $b_k = -\gamma_{m-k+1}$, $k=1, 2,\ldots,m$, where γ_k, $k=1,2,\ldots,m$ are defined by /4/. Then

$$P_m(\delta_t) = \prod_{k=1}^{m} (\delta_t - \mu_k) = (t\frac{d}{dt} - \mu_m)(t\frac{d}{dt} - \mu_{m-1}) \ldots (t\frac{d}{dt} - \mu_1) =$$

$$= (t^{\mu_m+1}\frac{d}{dt}t^{-\mu_m})(t^{\mu_{m-1}+1}\frac{d}{dt}t^{-\mu_{m-1}}) \ldots (t^{\mu_k+1}\frac{d}{dt}t^{-\mu_k}) \ldots (t^{\mu_1+1}\frac{d}{dt}t^{-\mu_1})$$

$$= t^{-\beta\gamma_1+1}\frac{d}{dt}t^{\beta\gamma_1-\beta\gamma_2+1}\frac{d}{dt}t^{\beta\gamma_2-\beta\gamma_3+1} \ldots \frac{d}{dt}t^{\beta\gamma_m} =$$

$$= t^{\beta+\alpha_0}\frac{d}{dt}t^{\alpha_1} \ldots \frac{d}{dt}t^{\alpha_m} = t^{\beta} B$$

and hence the general Bessel-type differential operator /1/ can be rewritten in the form

$$/8/ \qquad B = t^{-\beta} P_m(\delta_t) = t^{-\beta}\prod_{k=1}^{m} (\delta_t + \beta\gamma_{m-k+1}) .$$

Thus /7'/ becomes

$$B\, y(\lambda t) = \lambda^{\beta} y(\lambda t)$$

which is equation /5/ . The theorem is proved.

<u>Remark.</u> The hyper-Bessel functions of P. Delerue ${}_0F_{m-1}$ (see O. Mariček [5] ,p.81) :

$$G^{1,0}_{0,m}\left[-(\lambda t)^{\beta} \,\middle|\, -\gamma_{m-k+1}, -\gamma_m, -\gamma_{m-1}, \ldots, -\gamma_{m-k+2}, -\gamma_{m-k}, \ldots, -\gamma_1\right]$$

$$= \prod_{\substack{h=1\\ h\neq m-k+1}}^{m} \Gamma^{-1}(1-\gamma_{m-k+1}+\gamma_h)\,(\lambda t)^{-\beta\gamma_{m-k+1}}\; {}_0F_{m-1}((1-\gamma_{m-k+1}+\gamma_h^{*}); (\lambda t)^{\beta}),$$

$$k=1,2,\ldots,m$$

form the fundamental system for the equation /5/ in the neighbourhood of the regular singular point $t = 0$. Moreover any of the m functions $G^{s,0}_{0,m}\left[(-1)^s \frac{t^{\beta}}{(-\beta)^m} \,\middle|\, (-\gamma_{m-k+1})\right]$, $1 \leq s \leq m$ satisfies the same equation /5/. That's why Meijer's G-functions with $q=m$, $n=0$ play a natural role in problems concerning Bessel-type operators.

Usually the operator B is considered not directly, but by means of a right inverse operator of it. This is the linear integral operator

$$/9/ \qquad Lf(t) = \frac{t}{\beta^m}\int_0^1 \ldots \int_0^1 f(t(t_1\ldots t_m)^{\frac{1}{\beta}}) \prod_{k=1}^{m} t^{\gamma_k}\, dt_1\ldots dt_2 \;,$$

defined in a space C_α, where $\{\gamma_k\}$ are the parameters /4/, $\beta = m-(\alpha_0+\alpha_1+\ldots+\alpha_m)$, $\alpha = \max_{0\le k\le m-1}(\alpha_0+\alpha_1+\ldots+\alpha_k-k-1)$, $C_\alpha = \{ f(t)=t^p\tilde{f}(t)$; $p>\alpha$, $\tilde{f}\in C[0,\infty)\}$.

If we suppose $\beta>0$, $\gamma_1<\gamma_2<\ldots<\gamma_m<\gamma_1+1$, then $\alpha=-\beta(\gamma_1+1)$ and $L : C_\alpha \longrightarrow C_\alpha$.

As it is proved by I. Dimovski [3] there exists a Volterra transformation of the first kind $\varphi : C_\alpha \longrightarrow C_{-1}$ of the form

$$/10/ \quad \varphi f(t) = \left[t^{m(\gamma_m+1)-1} / \prod_{k=1}^{m-1}\Gamma(\lambda_k)\right]\int_0^1\ldots\int_0^1 \prod_{k=1}^{m-1}\left[(1-t_k)^{\lambda_k-1} t_k^{\gamma_k}\right.$$

$$\left. f(t^{m/\beta}(t_1\ldots t_m)^{1/\beta})\right] dt_1\ldots dt_m$$

with $\lambda_k=\gamma_m-\gamma_k+\frac{k}{m}$, $k=1,\ldots,m-1$, which is a transmutation operator from L to the operator of multiple integration ℓ^m. The similarity relation $\varphi L = (\frac{m}{\beta}\ell)^m$ allows to transfer the results from the classical Heaviside-Mikusiński calculus to an operational calculus for Bessel-type operators. Here we give a new representation of this transmutation operator in terms of generalized Riemann-Liouville operators of fractional integration, defined as

$$/11/ \quad \mathcal{R}_{\beta,m}^{\{\gamma_k\},\{\delta_k\}} f(t) = t^{-\beta}\int_0^t G_{m,m}^{m,0}\left[\frac{\tau^\beta}{t^\beta}\middle|\begin{matrix}\gamma_k+\delta_k\\ \gamma_k\end{matrix}\right] f(\tau)\, d(\tau^\beta) ,$$

where $m,\beta,\{\gamma_k\}$ are the parameters of the operator B and the real numbers $\{\delta_k\}$, $k=1,\ldots,m$ play the role of m-dimensional order of fractional integration. In [6] it is proved the following

<u>Theorem 2.</u> For functions of the space C_α the transmutation operator φ defined by /10/ can be represented in the form

$$/12/ \quad \varphi f(t) = t^{m(\gamma_m+1)-1}\int_0^1 G_{m-1,m-1}^{m-1,0}\left[\tau\middle|\begin{matrix}\frac{k}{m}\\ \gamma_k-\gamma_m\end{matrix}\right]\tau^{\gamma_m} f(t^{\frac{m}{\beta}}\tau^{\frac{1}{\beta}})\, d\tau ,$$

i.e.

$$/12'/ \quad f(t^{\frac{\beta}{m}}) = t^{\beta(\gamma_m+\frac{m-1}{m})}\mathcal{R}_{\beta,m-1}^{\{\gamma_k\},\{\lambda_k\}} f(t) .$$

Conversely, if $\varphi f(t) = F(t)$, then the following inversion formula holds:

$$/13/ \quad f(t) = t^{-\beta(\gamma_m+\frac{m-1}{m})} \int_0^1 G^{m-1,0}_{m-1,m-1}\left[\tau \middle| \begin{matrix} \gamma_k-\gamma_m \\ \frac{k}{m} \end{matrix}\right] \tau^{\frac{1}{m}-1} F(t^{\frac{\beta}{m}} \tau^{\frac{1}{m}})\, d\tau =$$

$$= t^{-\beta(\gamma_m+\frac{m-1}{m})} \mathcal{R}^{\left\{\frac{k-m+1}{m}\right\},\ \{-\lambda_k\}}_{\beta,\ m-1} F(t^{\frac{\beta}{m}}) .$$

Using Theorem 2 we can give explicit representations for the convolutions of L in C_α ,for the operator L and its integer and fractional powers (see [6]). They are simpler then those of [1], [2], [3].

The transform approach to the Bessel-type operational calculus is based on the modification of an integral transformation of N. Obrechkoff (1958) ,defined as (see [2], [3]) :

$$/14/ \quad F(z) = \mathcal{O}\left\{ f(t)\ ;\ z \right\} = \beta \int_0^\infty K\left[(zt)^\beta\right] t^{\beta(\gamma_m+1)-1} f(t)dt$$

with the kernel function

$$/15/ \quad K(z) = \int_0^\infty \cdots \int_0^\infty \exp(-u_1 - \ldots - u_{m-1} - \frac{z}{u_1\ldots u_{m-1}})$$

$$\prod_{k=1}^{m-1} u_k^{\gamma_k-\gamma_m-1} \, du_1 \ldots du_{m-1} \quad .$$

We consider this generalization of the Laplace transform in the space Ω of the functions of C_α ,which are $O(\exp \lambda t^{\frac{\beta}{m}})$ for $t \to \infty$. Some useful operational properties of /14/ have been given in [2], [3], [7]. Now we propose a new definition of the same transformation, viz.

<u>Theorem 3.</u> Let the function $f(t) \in \Omega$,then the Obrechkoff transform /14/ of it can be rewriten in the form of the following one-dimensional integral

$$/16/ \quad \mathcal{O}\left\{ f(t)\ ;\ z \right\} = \beta z^{-\beta(\gamma_m+1)+1} \int_0^\infty G^{m,0}_{0,m}\left[(zt)^\beta \middle| (1+\gamma_k-\tfrac{1}{\beta})\right] f(t)dt.$$

<u>Remark.</u> Analogously to Theorem 1 ,it is easy to prove that the kernel function $y(z,t) = z^{-\beta(\gamma_m+1)+1} G^{m,0}_{0,m}\left[(zt)^\beta \middle| (1+\gamma_k-\frac{1}{\beta})\right]$ of /16/ satisfies the equation $B^* y(z,t) = (-\beta)^m z^\beta y(z,t)$,where

$B^* = t^{\alpha_m}\frac{d}{dt}t^{\alpha_{m-1}} \dots \frac{d}{dt}t^{\alpha_0}$ .Thus using integration by parts we receive that the following basic operational property is satisfied :

$$/17/ \quad \mathcal{O}\left\{L^{\lambda} f(t) \; ; \; z\right\} = \left\{1/(\beta^m z^{\beta})^{\lambda}\right\}.\mathcal{O}\left\{f(t) \; ; \; z\right\} .$$

Using /13/ and the relation between the usual Laplace transform and the Obrechkoff transform ,given in [7] :

$$/18/ \quad \mathcal{L}\left\{\varphi f(t) \; ; \; z\right\} = (2\pi i)^{\frac{1-m}{2}} m^{-\frac{1}{2}}.\mathcal{O}\left\{f(t) \; ; \; \left(\frac{z}{m}\right)^{\frac{m}{\beta}}\right\} ,$$

we get a complex inversion formula for /16/ , viz.

<u>Theorem 4.</u> Let $f(t) \in \Omega$ be a continuous function and $F(z)$ be the Obrechkoff transform of it , then

$$/19/ \quad f(t^{\frac{1}{\beta}}) = \frac{1}{(2\pi)^m i} \int_{\sigma-i\infty}^{\sigma+i\infty} G^{m,0}_{0,m}\left[\left(-\frac{z}{m}\right)^m t \;\middle|\; \left(\frac{m-k+1}{m}+\lambda_k\right)\right] F\left(\frac{z}{m}\right)^{\frac{m}{\beta}} dz ,$$

where σ is a sufficiently large constant .

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