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DOMINATED APPROXIMATION OF CONVEX FUNCTIONS BY POSITIVE LINEAR OPERATORS

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1. Introduction. Let $C[a,b]$ $(-\infty<a<b<+\infty)$ denotes the space of all continuous functions $f:[a,b]\to R$, and $K[a,b]$ denotes the cone of functions convex on (a,b), and also continuous from the right at $x=a$ and from the left at $x=b$. As usual, we say that f is convex on (a,b) if for every $x,y\in(a,b)$ and $t\in(0,1)$ the inequality of JENSEN $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ holds. Further, let $(L_p)(p\in P)$ is a family of linear operators $L_p: C[a,b]\to C[a,b]$, where P is an index-set with infinite but arbitrary cadinality, and let p_o is an accumulation point of P. We shall write $L_pf\ddagger f$ when $p\to p_o$ on $[a,b]$ if $\lim_{p\to p_o}\|L_pf-f\|_C=0$. Also, we shall say that L_pf approximates f dominatedly (or from above) if $L_pf\ddagger f$ and $L_pf\geq f$ for every $p\in P$.

It is known (see expositure paper [6]) that the sequence of BERNSTEIN polynomials

$$(1)\qquad (B_nf)(x) = \sum_{k=o}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} \qquad (n\in N)$$

converging dominately on $[0,1]$ to every f which is convex on $[0,1]$. This is also valid for sequence of MEYER-KÖNIG-ZELLER operators [5,4]

$$(2)\qquad (M_nf)(x) = (1-x)^{n+1}\sum_{k=o}^{+\infty} f(\frac{k}{n+k})\binom{n+k}{k}x^k \qquad (n\in N),$$

i.e. $M_nf\downarrow\downarrow f$ for $n\to+\infty$ and $f\in K[0,1]$, and so forth.

Begining from the theorems of KOROVKIN [2], and the theorem of VASIĆ--LACKOVIĆ [7], we shall try to describe all families of positive linear operators $(L_p)(p\in P)$ providing dominated convergence of L_pf toward convex functions on finite interval $[a,b]$.

2. Main results. At the first, we shall prove an elementary lemma.

Lemma 1. The family $(L_p)(p \in P)$ of linear operators $L_p: C[a,b] \to C[a,b]$ converging uniformly to f when $p \to p_o$ (p_o is an accumulation point of the index-set P) if L_p satisfyes the following conditions:

(3) $$L_p e_p = e_o + \alpha_p, \quad L_p e_1 = e_1 + \beta_p, \quad L_p e_2 = e_2 + \gamma_p \quad \text{on } [a,b],$$

where $e_i(t) = t^i$ $(i=0,1,2)$ and $\alpha_p(x)$, $\beta_p(x)$ and $\gamma_p(x)$ tend to zero, uniformly on $[a,b]$, when $p \to p_o$.

Proof. Follows directly from the theorem of KOROVKIN [2], and the statement of HEINE-BOREL. Namely, if $p \to p_o$, we can choose an arbitrary sequence $(p_n)(n \in N)$ so that $\lim p_n = p_o$. So, we can apply the KOROVKIN theorem to L_{p_n}, which completes the proof.

Also, we need the following statement proved by P.M. VASIĆ and I.B. LACKOVIĆ [7]:

Lemma 2. Let us assume that every operators L_p: $C[a,b] \to C[a,b]$ is linear and continuous. Then, for every function f and every $p \in P$ the implication $f \in K[a,b] \Rightarrow L_p f \geqq 0$ hold if and only if

(4) $$L_p e_o = 0,$$

(5) $$L_p e_1 = 0,$$

(6) $$L_p e_c \geqq 0 \quad \text{for every } c \in [a,b],$$

hold, for all $p \in P$. For $c \in [a,b]$, the function $t \to e_c(t)$, $t \in [a,b]$ is defined by $e_c(t) = |t-c|$.

Now, we have

Theorem 1. Let the family $(L_p)(p \in P)$ of positive linear operators L_p mapping $C[a,b]$ into itself satisfy the conditions

(7) $$L_p e_o = e_o, \quad L_p e_1 = e_1, \quad L_p e_2 = e_2 + \gamma_p \quad \text{on} \quad [a,b],$$

where $\gamma_p \rightrightarrows 0$ on $[a,b]$ when $p \to p_o$, then for every $f \in K[a,b]$ we have $L_p f \downarrow\downarrow f$ on $[a,b]$ $(p \to p_o$, p_o is an accumulation point of index-set $P)$.

Proof. According with lemma 1, we have that $L_p f \rightrightarrows f$ $(p \to p_o)$ for every $f \in K[a,b]$ because $K[a,b] \in C[a,b]$. If we put.

(8) $$E_p f = L_p f - f, \qquad p \in P.$$

It is obvious that E_p is a linear operator for every $p \in P$. Let's prove that E_p is continuous. Namely, let the sequence $(g_n)(n \in N)$ converging uniformly to $g \in C[a,b]$. Then, for every $p \in P$, $E_p(g_n) = L_p(g_n) - g_n$, and when $n \to +\infty$ we have $L_p(g_n) = L_p(g)$ so $E_p(g_n) \rightrightarrows E_p(g)$ $(n \to +\infty)$. Further, we have $E_p e_o = 0$ and $E_p e_1 = 0$, in virtue of (4) and (5). Let's show that the condition (6) is also fulfiled. For every $c \in [a,b]$ and $p \in P$, we have

$$(9) \qquad (E_p|t-c|)(x) = (L_p|t-c|)(x) - |x-c|.$$

According with linearity of L_p and (4) and (5) we get $|(L_p(t-c))(x)| = |(L_p(e_1 - ce_o))(x)| = |(L_p e_1)(x) - c(L_p e_o)(x)| = |e_1(x) - ce_o(x)| = |x-c|$.
But, from positivity of L_p, we have $|L_p f| \leqq L_p|f|$, or $(L_p|t-c|)(x) \geqq |(L_p(t-c))(x)|$ which, together with (9) gives

$$(10) \qquad (L_p|t-c|)(x) \geqq |x-c|$$

for every $c \in [a,b]$ and $p \in P$. The inequality (10) does mean nothing else but $E_p e_c \geqq 0$, for every $c \in [a,b]$, $p \in P$. Then, in virtue of lemma 2, we have that $E_p f \geqq 0$ for every $f \in K[a,b]$ i.e. $L_p f \geqq f$ for every $f \in K[a,b]$. Then, we have $L_p f \downarrow\downarrow f$ $(p \to p_o)$. The proof is completed.

The next theorem gives an estimation of fast of convergence.

Theorem 2. Let the family $(L_p)(p \in P)$ of positive linear operators satisfying conditions of theorem (7). Then

(a) $\gamma_p \downarrow\downarrow 0$ *when* $p \to p_o$,

(b) $(L_p f)(x) - f(x) \leq 2\omega_f(\sqrt{\gamma_p(x)})$,

for every $f \in K[a,b]$, where ω_f is modulus of continuity defined as

$$\omega_f(h) = \sup_{0 \leq t \leq h} \|f(x+t) - f(x)\|_C \quad (x \in [a,b-t],\ 0 \leq h \leq b-a).$$

Proof. (a) The function γ_p from (7) uniformly tends to zero as we supposed. But, $\gamma_p \geqq 0$ for every $p \in P$, because of

$$0 \leq (L_p(e_1 - xe_o)^2)(x) = (L_p(e_1^2 - 2x\, e_o + x^2 e_o^2))(x) = (L_p e_1^2)(x)$$
$$- 2x(L_p e_o)(x) + x^2(L_p e_o^2)(x) = (L_p e_2)(x) - 2xe_o(x) + x^2 e_o(x)$$
$$= e_2(x) + \gamma_p(x) - 2xe_1(x) + x^2 e_o(x) = \gamma_p(x).$$

(b) On the basis of theorem of R. MAMEDOV and its generalizations due by O. SHISHA and B. MOND [1, p.28-29], for every $f \in C[a,b]$ we have $|(L_p f)(x)-f(x)| = |f(x)||e_o(x)-(L_p e_o)(x)|+((L_p e_o)(x)+(L_p e_o)^{1/2}(x))\omega_f(\alpha_p(x))$ where $\alpha_p{}^2(x) = (L_p(t-x)^2)(x)$. But, as we shown, $\gamma_p(x) \geq 0$ and $L_p f \geq f$, we have the estimation given in (b).

3. Examples. In this section we give a number of examples.

Ex. 1. Iterated BERNSTEIN operators. Operators B_n, given by (1) are positive, linear and $B_n e_o = e_o$, $B_n e_1 = e_1$, $B_n e_2 = e_2 + \gamma_n$, where $\gamma_n(x) = \frac{1}{n} x(1-x)$ ([3]). Also, it is known that $B_n f$, for fixed n, is a kind of interpolating polynomials which have not an idempotence property, i.e. $B_n(B_n f) \neq B_n f$, as LAGRANGE polynomials do have. So we may construct a nontrivialy iteration

$$B_n^{(1)} = B_n, \quad B_n^{(m)} = B_n(B_n^{(m-1)}), \quad m=2,3,\dots .$$

Now, it is easy to show by induction that $B_n^{(m)} e_o = e_o$, $B_n^{(m)} e_1 = e_1$, $B_n^{(m)} e_2 = e_2 + \gamma_n^{(m)}$ where $\gamma_n^{(m)} = x(1-x)(1-(\frac{n-1}{n})^m)$. It is obvious that $\gamma_n^{(m)} \ddagger 0 \ (n \to +\infty)$ for every $m \in N$. Accordingly vith theorem 1, we have $B_n^{(m)} f \downarrow\downarrow f$ for every function f convex on $[0,1]$ and every $m \in N$. For $m=1$, we have the results of POLYA and SCHOENBERG, L. KOSMÁK, E. MOLDOVAN and others (See [6]).

Ex. 2. Iterated operators of MAYER-KÖNIG-ZELLER. If we put $M_n^{(1)} = M_n$ where M_n is given by (2), $M_n^{(m)} = M_n(M_n^{(m-1)})$, $m=2,3,\dots$, we have iterated operators of MEYER-KÖNIG--ZELLER. Of course, $M_n : C[0,1] \to C[0,1)$ and $M_n^{(m)} : C[0,1) \to C[0,1)$. The operators M_n are linear and continuous, so the $M_n^{(m)}$ are too. It is known that $M_n e_1 = e_1$ and $M_n e_2 = e_2 + \gamma_n + R_n$, where $\gamma_n = \frac{1}{n} x(1-x)^2$, and R_n is estimated by (LUPAȘ-MÜLLER [4])

$$|R_n(x)| = \frac{8}{27n(n-1)} = O(\frac{1}{n}) \quad \text{when} \quad n \to +\infty .$$

So, for every finite $m \in N$, we have $M_n^{(m)} e_o = e_o$, $M_n^{(m)} e_1 = e_1$ and

$$M_n^{(m)} e_2 = e_2 + \gamma_n^{(m)} + m\, o(\frac{1}{n})$$

where $\gamma_n^{(m)} = \sum_{k=o}^{m-1} M_n^{(k)} \gamma_n$ and $M_n^{(0)}$ is identical operator. The last formula easy obtains by induction. As $\gamma_n(x) \ddagger 0 \ (n \to +\infty)$ then also $M_n^{(k)} \gamma_n \ddagger 0 \ (n \to +\infty)$ in virtue of continuity of $M_n^{(k)}$ for every $k \in N$. So we have $M_n^{(m)} f \downarrow\downarrow f$ for every $f \in K[0,1)$ and $m \in N$.

Ex. 3. Iterated STEKLOV operators. For $f\in C[a,b]$ we regard the operator

$$(S_h f)(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt \qquad (a \leq x-h < x+h \leq b).$$

which sometimes called STEKLOV function, and having an important application in meny area of analysis. It is easy to see that $S_h e_o = e_o$, $S_h e_1 = e_1$ and $S_h e_2 = e_2 + h^2/3$. So, in virtue of theorem 1, we have $S_h f \downarrow\downarrow f$ when $h\to 0+$ for every $f\in K[a,b]$.
Now, we put

$$S_h^{(1)} = S_h, \quad S_h^{(m)} = S_h(S_h^{(m-1)}), m=2,3,\dots .$$

Without any difficulties, we can obtain that $S_h^{(m)} e_o = e_o$, $S_h^{(m)} e_1 = e_1$ and $S_h^{(m)} e_2 = e_2 + m(h^2/3)$ for every $m\in N$. As, for every finite m $mh^2\to 0$ when $h\to 0$, we have also that $S_h^{(m)} \downarrow\downarrow f$ when $h\to 0$ and $f\in K[a,b]$.

Indeed $S_h^{(m)}$, where m is fixed, ve can observe the sequence $(S_{h_m}^{(m)})(m\in N)$. Similary as above, $S_{h_m}^{(m)} e_2 = e_2 + m(h_m^2/3)$. Then, if $m\, h_m^2 \to 0 (m\to +\infty)$ we have that $S_{h_m}^{(m)} f\downarrow\downarrow f$ $(m\to +\infty)$ for arbitrary convex function f on $[a,b]$.

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