КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 468—472

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
Sofia, 1984, p.468-472

ON THE BOUNDEDNESS OF RIESZ POTENTIALS AND FRACTIONAL MAXIMAL FUNCTIONS IN WEIGHTED ORLICZ SPACES

Vachtang M. Kokilashvili and Miroslav Krbec

1. Introduction. This paper deals with properties of the Riesz operators

$$(1.1)\qquad (T_\gamma f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy , \quad 0 < \gamma < n ,$$

and fractional order maximal functions

$$(1.2)\qquad (M_\gamma f)(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\gamma/n}} \int_Q |f(y)|dy , \quad 0 < \gamma < n ,$$

(the supremum being taken over all cubes $Q \subset \mathbb{R}^n$ containing x and with sides parallel to coordinate axes) in weighed Orlicz spaces

$$L_\Phi(\rho) = \{f;\ ||f||_{L_\Phi(\rho)} = \inf\{\lambda > 0;\ \int_{\mathbb{R}^n} \Phi(\lambda^{-1} f(x))\rho(x)\, dx \leq 1\} < \infty\} .$$

Let us briefly summarize important results; the boundedness of the operator (1.1) and/or (1.2) in L_p-spaces was investigated by Hardy and Littlewood in [3] (n = 1, and including power weights, too) followed by the fundamental Sobolev theorem [11] and then there is a series of theorems on weighted spaces (weights of power or somewhat more general type) [12], [7], [13], ... , and, finally, a complete answer to the weighted problem in L_p-spaces is due to Muckenhoupt and Wheeden [9] who fully characterized all admissible weights in terms of the so-called A_p-*classes* (or *the Muckenhoupt classes*) consisting of all a.e. positive functions $\rho \in L_{1,loc}$ satisfying

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \rho(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q (\rho(x))^{-1/(p-1)}\, dx\right)^{p-1} < \infty$$

(again, the supremum over all cubes Q with sides parallel to coordinate axes). Recently, Kerman and Torchinsky [5] proved that the (non--fractional) maximal function is bounded in $L_\Phi(\rho)$ iff $\rho \in A_{i(\Phi)}$ where

$$i(\Phi) = \lim_{\lambda \to 0} \frac{1}{\log \lambda} \log \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)} \tag{1.3}$$

is *the lower index of* Φ.

Here, we are going to study operators of the type (1.1) and (1.2) acting from a weighted Orlicz space into another one. Somewhat surprisingly - as in [5] - we obtain a condition in the A_p-classes language.

We shall suppose that a reader is familiar with basic facts from the theory of Orlicz spaces. We work with Young functions in sense of [4]. The number

$$I(\Phi) = \lim_{\lambda \to \infty} \frac{1}{\log \lambda} \log \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)} \tag{1.4}$$

is *the upper index of* Φ (cf. (1.3); see [8], [1]). A Young function Φ such that $I(\Phi) < \infty$ is said to satisfy the Δ_2-condition.

2. The main result.

2.1. Theorem. *Let* Φ_1 *and* Φ_2 *be Young functions such that* $1 < i(\Phi_1) = p \le I(\Phi_1) < \infty$, $1 < i(\Phi_2) = q \le I(\Phi_2) < \infty$. *Let* ρ *be a weight in* $\mathbb{R}^n$, $0 < \gamma < n$, *and* $\Phi_2^{-1}(t) \sim t^{-\gamma/n}\Phi_1^{-1}(t)$. *Then the following conditions are equivalent*

(i) $||T_\gamma(f\rho^{\gamma/n})||_{L_{\Phi_2}(\rho)} \le c||f||_{L_{\Phi_1}(\rho)}$,

(ii) $||M_\gamma(f\rho^{\gamma/n})||_{L_{\Phi_2}(\rho)} \le c||f||_{L_{\Phi_1}(\rho)}$,

(iii) $\rho \in A_s$ *where* $s = 1 + q/p'$ $(p' = \frac{p}{p-1})$.

Sketch of the proof. The implication (i) $\Rightarrow$ (ii) follows directly from the well-known estimate

$$(M_\gamma g)(x) \le c(T_\gamma|g|)(x) .$$

Now let us suppose that (ii) holds with $\varepsilon\rho$ instead of ρ for all $\varepsilon > 0$. We prove that $\varepsilon\rho \in A_s$ (which is, of course, equivalent to $\rho \in A_s$). This will be done in two steps.

Step 1. Let Ψ_1 be a Young function complementary to Φ_1, $\varepsilon > 0$, $Q \subset \mathbb{R}^n$ a cube, and $y \in Q$. Let f_0 be an a.e. non-negative function supported in Q, with the $L_{\Phi_1}(\varepsilon\rho)$-norm equal 1, and

$||f_0(\varepsilon\rho)^{\gamma/n}||_{L_1(Q)} = ||\chi_Q/(\varepsilon\rho)^{1-\gamma/n}||_{L_{\Psi_1}(\varepsilon\rho)}$. From the definition of M_γ we have

$$(2.1) \qquad \left(M\left(f_0(\varepsilon\rho)^{\gamma/n}\right)\right)(y) \geq \frac{1}{|Q|^{1-\gamma/n}} \left|\left|\frac{\chi_Q}{(\varepsilon\rho)^{1-\gamma/n}}\right|\right|_{L_{\Psi_1}(\varepsilon\rho)} \chi_Q(y) ,$$

and, using (ii) and taking $L_{\Phi_2}(\varepsilon\rho)$-norm on both sides of (2.1) we get

$$(2.2) \qquad \left|\left|\frac{\chi_Q}{(\varepsilon\rho)^{1-\gamma/n}}\right|\right|_{L_{\Psi_1}(\varepsilon\rho)} ||\chi_Q||_{L_{\Phi_2}(\varepsilon\rho)} \leq c|Q|^{1-\gamma/n} .$$

Step 2. We reduce (2.2) to the case described in the following

Proposition ([5]). Let Φ and Ψ be complementary Young functions both satisfying the Δ_2-condition. If $||\chi_Q/\varepsilon\rho||_{L_\Psi(\varepsilon\rho)} ||\chi_Q||_{L_\Phi(\varepsilon\rho)} \leq$ $\leq c|Q|$ for all cubes $Q \subset \mathbb{R}^n$ and $\varepsilon > 0$ then $\rho \in A_{i(\Phi)}$.

Now set $\beta = n/(n-\gamma)$, $\Phi_3^{-1}(t) = \left(\Phi_2^{-1}(t)\right)^\beta$, and $\Phi_3(t) =$ $= \Psi_1(t^{1/\beta})$. One can verify that Φ and Ψ form a complementary couple of Young functions, and that $||\chi_Q||^\beta_{L_{\Phi_2}(\varepsilon\rho)} = ||\chi_Q||_{L_{\Phi_3}(\varepsilon\rho)}$, $||\chi_Q/(\varepsilon\rho)^{1-\gamma/n}||^\beta_{L_{\Psi_1}(\varepsilon\rho)} = ||\chi_Q/\varepsilon\rho||_{L_{\Psi_3}(\varepsilon\rho)}$.

Hence (2.2) yields

$$\left|\left|\frac{\chi_Q}{\varepsilon\rho}\right|\right|_{L_{\Psi_3}(\varepsilon\rho)} ||\chi_Q||_{L_{\Phi_3}(\varepsilon\rho)} \leq c|Q|$$

so that $\rho \in A_{i(\Phi_3)}$ in accordance with the preceding Proposition. Because $i(\Phi_3) = 1/I(\Phi_3^{-1}) = 1/\beta I(\Phi_2^{-1}) = q(1 - \gamma/n) = s$, (iii) follows.

It remains to prove that (iii) $\Rightarrow$ (i). A detailed proof requires some particular interpolation results so that we refer to [6] and [2] and just notice that if Φ is a Young function with $1 < i(\Phi)$, $I(\Phi) <$ $< \infty$ then $L_\Phi(\rho)$ can be obtained by interpolation of any couple $\left(L_{r_1}(\rho), L_{r_2}(\rho)\right)$ where $1 < r_1 < i(\Phi)$, $I(\Phi) < r_2 < \infty$. So let $\rho \in$ $\in A_s$. A known fact is that then $\rho \in A_{s_1}$ for some $s_1 < s$, and $\rho \in$ $\in A_{\tilde{s}}$ for all $\tilde{s} > s_1$. As

$$\frac{1}{p} = \frac{\gamma}{n} + \frac{1}{s}\left(1 - \frac{\gamma}{n}\right) , \text{ and } \frac{1}{q} = \frac{1}{s}\left(1 - \frac{\gamma}{n}\right) ,$$

choosing s_1 sufficiently close to s , s_2 large enough, and putting

$$\frac{1}{p_i} = \frac{\gamma}{n} + \frac{1}{s_i}, \text{ and } \frac{1}{q_i} = \frac{1}{s_i}\left(1 - \frac{\gamma}{n}\right), \quad i = 1,2,$$

we have $1 < p_1 < p \leq I(\Phi_1) < p_2$, $1 < q_1 < q \leq I(\Phi_2) < q_2$. By interpolation procedure applied to the Muckenhoupt-Wheeden result [9]

$$||T_\gamma(f\rho^{\gamma/n})||_{L_{q_i}(\rho)} \leq c||f||_{L_{p_i}(\rho)}, \quad i = 1,2,$$

(i) follows.

Remarks. 1. A complete detailed proof together with applications to Sobolev-Orlicz weighted spaces can be found in [6].

2. A routine plays the major role in a generalization of Theorem 2.1 to anisotropic fractional order maximal functions.

References

1. D. W. Boyd. Indeces of function spaces and their relationship to interpolation. Canad. J. Math. 21 (1969), 1245-1254.
2. J. Gustavsson and J. Peetre. Interpolation of Orlicz spaces. Studia Math. 60 (1977), 33-59.
3. G. H. Hardy and J. E. Littlewood. Some properties of fractional integrals I. Math. Z. 27 (1928), 565-606.
4. M. A. Krasnosel'skiǐ and Ja. B. Rutickiǐ. Convex functions and Orlicz spaces. Noordhoff, Groningen 1961.
5. R. A. Kerman and A. Torchinsky. Integral inequalities with weights for the Hardy maximal function. Studia Math. 71 (1982), 277-284.
6. V. M. Kokilashvili and M. Krbec. Weighted norm inequalities for fractional order maximal functions and Riesz potentials in Orlicz spaces. Preprint, Math. Inst. Czech. Acad. Sci. 5/1984.
7. P. I. Lizorkin. Multipliers of Fourier integrals and bounds on convolution in spaces with mixed norms. Applications (Russian). Izv. Akad. Nauk, Ser. Mat. 34 (1970), 225-255.
8. W. Matuszewska and W. Orlicz. On certain properties of ϕ-functions. Bull. Acad. Polon. Sci. 7 (1960), 439-443.
9. B. Muckenhoupt and R. L. Wheeden. Weighted norm inequalities for fractional integrals. Trans. Amer. Math. Soc. 192 (1974), 261-274.
10. J. Musielak. Orlicz spaces and modular spaces. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1983.
11. S. L. Sobolev. On a theorem in functional analysis (Russian). Mat. Sb. 4 (46) (1938), 471-497. (Amer. Math. Soc. Transl. 2 (34) (1963), 39-68.)
12. E. M. Stein and G. Weiss. Fractional integrals on n-dimensional Euclidean space. J. Math. Mech. 7 (1958), 503-514.
13. T. Walsh. On weighted norm inequalities for fractional and singular integrals. Canad. J. Math. 23 (1971), 907-928.

(V. M. Kokilashvili)
Mathematical Institute
Georg. Acad. Sci.
Z. Ruchadze 1
Tbilisi 380093
USSR

(M. Krbec)
Mathematical Institute
Czech. Acad. Sci.
Žitná 25
115 67 Prague 1
Czechoslovakia