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ANISOTROPIC MAXIMAL INEQUALITIES WITH WEIGHTS

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1. Introduction. By a *weight* we shall mean a measurable function which is non-negative and finite a.e. in $\mathbb{R}^n$. A weight w generates the measure μ_w :

$$\mu_w e = \int_e w(x)\, dx , \quad e \subset \mathbb{R}^n \text{ measurable.}$$

The Lebesgue measure of a set e will be denoted by $|e|$.

Let $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_i > 0$ for $i = 1,\dots,n$ be fixed. For $x \in \mathbb{R}^n$ and $t > 0$ we put

$$E(x,t) = \{z \in \mathbb{R}^n ;\ |z_i - x_i| < \tfrac{1}{2} t^{\alpha_i} ,\quad i = 1,\dots,n\}$$

and by $\mathbb{E} = \mathbb{E}(\alpha)$ we denote the *one-parametric class of all parallelepipeds* $E(x,t)$, $x \in \mathbb{R}^n$, $t > 0$.

To every measurable function $f : \mathbb{R}^n \to \mathbb{R}^1$ we adjoin the anisotropic maximal function

(1)
$$Mf(x) = \sup_{t>0} |E(x,t)|^{-1} \int_{E(x,t)} |f(z)|\, dz .$$

If $\alpha_1 = \dots = \alpha_n$, then Mf is the classical Hardy-Littlewood maximal function. For the operator M the anisotropic version of the lemma by C. Fefferman and E. M. Stein [3] can be proved:

Lemma 1. *Let* $1 < p < \infty$. *Then there exists a constant* $c > 0$ *such that the inequality*

$$\int_{\mathbb{R}^n} [Mf(x)]^p |g(x)|\, dx \le c \int_{\mathbb{R}^n} |f(x)|^p Mg(x)\, dx$$

holds for all $f, g \in L_{loc}(\mathbb{R}^n)$.

We introduce an anisotropic analogue of Muckenhoupt's classes A_p. Let $1 < p < \infty$. *A weight* w *belongs to the class* $A_p(\alpha)$ if there exists $c > 0$ such that

$$\left(|E|^{-1}\int_E w(z)\,dz\right)\left(|E|^{-1}\int_E w^{-\frac{1}{p-1}}(z)\,dz\right)^{p-1} \le c$$

for every $E \in E(\alpha)$. *A weight* w *is of the class* $A_1(\alpha)$ if there exists $c > 0$ such that

$$Mw(x) \le cw(x) \quad \text{a.e. in } \mathbb{R}^n ,$$

where M is defined by (1).

2. Vector-valued anisotropic maximal function. The well known Hardy-Littlewood theorem on the L^p boundedness of the classical maximal operator was generalized in several ways. C. Fefferman and E. M. Stein [3] proved the maximal inequality for ℓ^θ-valued functions. B. Muckenhoupt [11] characterized weights w for which the maximal operator is bounded in L^p_w (the Lebesgue space with the weight w). D. S. Kurtz [9] solved this problem for the anisotropic maximal operators. The case of L^θ-valued functions was investigated by the first author [5] and for ℓ^θ-valued functions it was done independently by K. F. Andersen and R. T. John [1].

We shall deal here with a vector-valued anisotropic maximal function, which is defined as follows: Let (Y,S,ν) be a σ-finite measure space and T be a σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$. On the σ-algebra $T \times S$ we define the measure λ as the product of the Lebesgue measure and of ν. To any λ-measurable function $f : \mathbb{R}^n \times Y \to \mathbb{R}^1$ we adjoin the *vector-valued anisotropic maximal function*

$$M_{(1)}f(x,y) = \sup_{t>0} |E(x,t)|^{-1} \int_{E(x,t)} |f(z,y)|\,dz , \quad x \in \mathbb{R}^n,\ y \in Y.$$

For the operator $M_{(1)}$ we state a weak type inequality:

Lemma 2. *If* $1 \le p \le \theta < \infty$, $\theta > 1$ *and* $w \in A_p(\alpha)$, *then there exists* $c > 0$ *such that*

$$\mu_w\left\{x \in \mathbb{R}^n;\ \left(\int_Y [M_{(1)}f(x,y)]^\theta\,d\nu\right)^{1/\theta} > s\right\} \le$$

$$\le cs^{-p}\int_{\mathbb{R}^n}\left(\int_Y |f(x,y)|^\theta\,d\nu\right)^{p/\theta} w(x)\,dx$$

for every $s > 0$ *and every* λ*-measurable function* f.

By means of this weak type inequality and of interpolation theorems a strong type inequality can be proved:

Theorem 1. *Let* $1 < p, \theta < \infty$. *Then the inequality*

$$\int_{\mathbb{R}^n} \left(\int_Y [M_{(1)} f(x,y)]^{\theta} \, d\nu\right)^{p/\theta} w(x)\, dx \le c \int_{\mathbb{R}^n} \left(\int_Y |f(x,y)|^{\theta} \, d\nu\right)^{p/\theta} w(x)\, dx$$

holds for every λ*-measurable function* f *with a constant* $c > 0$ *independent of* f *if and only if* $w \in A_p(\alpha)$.

Let us note, that even in the unweighted case Theorem 1 is more general then the result of C. Fefferman and E. M. Stein [3].

3. Maximal inequality with mixed norms. Let $\alpha^{(1)} \in \mathbb{R}^m$ and $\alpha^{(2)} \in \mathbb{R}^n$ be given vectors with positive components. Let M be the anisotropic maximal operator from (1) corresponding to the vector $\alpha = (\alpha^{(1)}, \alpha^{(2)}) \in \mathbb{R}^{m+n}$.

Theorem 2. *If* $1 < p_i < \infty$ *and* $w_i \in A_p(\alpha^{(i)})$, $i = 1,2$, *then the inequality*

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} [Mf(x,y)]^{p_1} w_1(x)\, dx\right)^{p_2/p_1} w_2(y)\, dy \le$$

$$\le c \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)|^{p_1} w_1(x)\, dx\right)^{p_2/p_1} w_2(y)\, dy$$

holds with a constant $c > 0$ *independent of* f.

It is a simple consequence of Theorem 1. The more difficult problem with general weights $w = w(x,y)$ (not of the form $w = w_1(x) w_2(y)$) is still unsolved.

4. Maximal inequality with two weights. In 1978 B. Muckenhoupt stated following two problems [12]: Let an integral operator T be given. *Characterize the weights* v *for which there exists a weight* w *such that* T *is a bounded mapping from* L^p_w *into* L^p_v. And, conversely, *characterize the weights* w *for which there exists a weight* v *with this property*. E.T.Sawyer [14], W. S. Young [15], A. E. Gatto and C. E. Gutiérrez [4], P. Koosis [8], L. Carleson and P. Jones [2] and J. L. Rubio de Francia [13] solved these problems for the Hardy-Littlewood maximal operator and for singular integral operators. Here we shall deal with the anisotropic maximal operator M defined by (1).

A. E. Gatto and C. E. Gutiérrez [4] introduced modificated maximal functions, whose anisotropic analogues we define as follows:

$$\bar{f}(x) = \sup_{t<\tau(x)} |E(x,t)|^{-1} \int_{E(x,t)} |f(y)|\, dy ,$$

$$\bar{\bar{f}}(x) = \sup_{\substack{t<2\tau(x)\\ E(z,t)\ni x}} |E(z,t)|^{-1} \int_{E(z,t)} |f(y)|\, dy ,$$

where $\tau(x) = \frac{1}{2} \left(\max_i 2|x_i|^{1/\alpha_i}\right)$, $x \in \mathbb{R}^n$. The following analogue of the lemma by C. Fefferman and E. M. Stein [3] (cf. Lemma 1 of this note) holds:

Lemma 3. *Let* $1 < p < \infty$. *Then there exists* $c > 0$

$$\int_{\mathbb{R}^n} [\bar{f}(x)]^p\, g(x)\, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p\, \bar{\bar{g}}(x)\, dx$$

for every measurable functions f , g , $0 < g(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$.

Let us define the "anisotropic norm" ρ in $\mathbb{R}^n$ by

$$\rho(x) = \left(\sum_{i=1}^{n} |x_i|^{2/\alpha_i}\right)^{|\alpha|/2n} ,$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The first of Muckenhoupt's problems mentioned above can be solved for the anisotropic maximal operator as follows:

Theorem 3. *Let* v *be a weight on* $\mathbb{R}^n$ *and* $1 < p < \infty$. *The following conditions are equivalent:*

(i) *There exists a positive weight* w *on* $\mathbb{R}^n$ *and a constant* $c > 0$ *such that the inequality*

$$\int_{\mathbb{R}^n} [Mf(x)]^p\, v(x)\, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx \tag{2}$$

holds for all $f \in L^p_w$.

(ii) $\displaystyle\int_{\mathbb{R}^n} v(x)[1 + \rho^n(x)]^{-p}\, dx < \infty$.

If the condition (ii) *is satisfied, then the weight* w *in* (i) *can be chosen in the form* $w(x) = \bar{\bar{v}}(x) + [1 + \rho^n(x)]^{\beta}$, $\beta > p - 1$.

The idea of the proof and of the construction of the weight w is due to A. E. Gatto and C. E. Guttiérrez [4]. While in Theorem 3 we obtained the weight w in an explicit form, we can solve the inverse problem in an existence form only, since the proof is based on the use of Maurey's factorization theorem [10]. In the proof we utilized essentially Lemma 2 of this note, too.

Theorem 4. *Let* w *be a positive weight on* $\mathbb{R}^n$ *and* $1 < p < \infty$. *Then the following conditions are equivalent:*

(i) *There exists a positive weight* v *on* $\mathbb{R}^n$ *such that the inequality* (2) *holds for all* $f \in L^p_w$ *with* c *independent of* f.

(ii) $w^{-\frac{1}{p-1}} \in L_{loc}(\mathbb{R}^n)$ *and*

$$\limsup_{t \to \infty} |E(0,t)|^{-1} \int_{E(0,t)} w^{-\frac{1}{p-1}}(x)\, dx < \infty .$$

The assertions of Sections 1 - 3 and 4 are proved in [6] and in [7], respectively.

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