КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 479—486

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
Sofia, 1984, p.479-486

SOMETHING ELSE ABOUT THE MEROMORPHIC CONTINUATION OF FUNCTIONS

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1. Let D be a domain in the complex plane $\bar{C}$, E a continuum in D, and G = D - E. It will be assumed throughout this paper, that the domain G is double-connected and regular wirh respect to the Dirichlet problem. We shall denote by h(z) (for $z \in G$) the harmonic measure on ∂E with respect to G. Let $F = \bar{C} - D$ and let C = C(E,F) and $\varrho = \varrho(G)$ be the capacity of the condenser (E,F) and the Riemann modulus of the domain G, respectively. It is known that $\varrho(G) = \exp\left[C(E,F)^{-1}\right]$. Denote by g(z) the harmonical conjugate function to h(z) in G. We set u(z) = $= \exp\left[-(h+ig)(z).C^{-1}\right]$. In the case being considered the function u is holomorphic and univalent in the domain G and represents it on the anulus $\{w,\ \varrho^{-1}<|w|<1\}$.

There exist sequences $\alpha = \{A_n\}_{n=1}^{\infty}$ and $\beta = \{B_n\}_{n=1}^{\infty}$, $A_n(z) = \prod_{k=1}^{n}(z-\alpha_{n,k})$, $B_n(z) = \prod_{k=1}^{n}(z-\beta_{n,k})$ with $\alpha_{n,k} \in E$, $\beta_{n,k} \in F$, $k=1,\dots,n; n=1,2,\dots$ such that

$$\lim_{n\to\infty} |A_n(z).B_n^{-1}(z)|^{1/n} = d.|u(z)|$$

uniformly in G (on each compact subset of G), where d is a positive constant; if $\beta_{n,k} = \infty$ for some pair (n,k), then the corresponding factor $1-z.\beta_{n,k}^{-1}$ is replaced by unity. Various constructions of such sequences α,β may be found, for example, in [1] and [2].

2. We consider a function f holomorphic on E ($f \in \mathcal{H}(E)$). The following construction is contained in [3] . Let (n,m) be an arbitrary

pair of nonnegative integers; we assume $n \geqq m$. There exist polynomials p and q, $q \not\equiv 0$, the degrees of which do not exceed n and m, respectively, such that

$$(2) \qquad (f.q.B_{n-m} - p).A_{n+m+1}^{-1} \in \mathcal{H}(E)$$

We define $R_{n,m} = R_{n,m}^{\alpha,\beta}(f) = p.(q.B_{n-m})^{-1}$

Although p and q are not uniquely determined by (2), for given n and m the construction yields unique rational function $R_{n,m}$ (the (n,m)- generalized Pade approximant for the function f corresponding to the pair (α,β)).

We set

$$R_{n,m} = P_{n,m}.(Q_{n,m}.B_{n-m})^{-1}$$

where $P_{n,m}$ and $Q_{n,m}$ have not a common divisor and $Q_{n,m}$ is monic. If $\deg Q_{n,m} = m$, then (2) holds with $p = P_{n,m}$, $q = Q_{n,m}$ and the rational function $R_{n,m}$ interpolates f at all the points $\alpha_{n+m+1,k}$, $k = 1,\dots,n$.

3. Let $\delta > \varrho^{-1}$, and let

$$D_\delta = \{z, \delta < |u(z)|\} \cup E, \quad \partial D_\delta = \Gamma_\delta, \quad G_\delta = D_\delta - E$$

For $\delta \geqq 1$, we assume $D_\delta = D$. The domain D_δ will be called canonical (for the geometric situation under consideration).

4. Fix a positive integer m ($m \in \mathbb{N}$). For each function f, holomorphic on E, we define $\mathcal{D}_m = \mathcal{D}_m(f)$ as the largest canonical domain, where f admits a continuation to a function meromorphic in $\mathcal{D}_m$ and having not more than m poles there (the poles are counted with regard to their multiplicities).

5. The following theorem is found in [3]:

THEOREM 1: Let f be holomorphic on E, $m \in \mathbb{N}$ be fixed and (α,β) be given as in (1). Then the sequence $\{R_{n,m}\}$ converges as $n \to \infty$ to the function f in capacity on each compact subset of $\mathcal{D}_m$.

Denote by $\nu(m) = \nu_f(m)$ the numbers of the poles of f in $\mathcal{D}_m$. It follows from Theorem 1 that

THEOREM 2: If $\nu(m) = m$, then $R_{n,m} \to f$ as $n\to\infty$ uniformly on each compact subset of $\mathcal{D}'_m = \mathcal{D}_m - \{\gamma_k\}_{k=1}^{m}$, where $\gamma_k, k=1,\dots,m$ are the poles of f in $\mathcal{D}_m$ and

a) $\deg Q_{n,m} = m$ for all n sufficiently large;

b) the sequence $\{Q_{n,m}\}$ converges as $n\to\infty$ to $Q(z) = \prod_{k=1}^{m}(z-\gamma_k)$ with the speed of a geometric progression, according to the relation

$$\limsup_{n\to\infty} \| Q_{n,m} - Q \|^{1/n} \leq R^{-1} \cdot \max_{1\leq k\leq m} | u(\gamma_k) | , \text{ where } \Gamma_R = \partial\mathcal{D}_m$$

(the norm is understood in the space of coefficients);

Detailed explanations of these results are to be found in [3].

The following theorem which reflects the preceding assertion for some classes (α,β) has been proved by the author (see [4]):

THEOREM 3: Let $f\in\mathcal{H}(E)$, let α, β be given such that (1) is valid and such that $A_n(z) = \prod_{k=1}^{n}(z-\alpha_k)$, $B_n(z) = \prod_{k=1}^{n}(z-\beta_k)$, $\alpha_n\in E$, $\beta_n\in F$, $n=1,2,\dots$ If there exists a polynomial $Q(z) = \prod_{k=1}^{m}(z-\gamma_k)$, $\gamma_k\in G$, $k=1,\dots,m$ such that for some q, $q<1$

(3) $$\limsup_{n\to\infty} \| Q_{n,m} - Q \|^{1/n} \leq q$$

then $\mathcal{D}_m \supseteq D_\sigma$, where $\sigma = q^{-1}\max_{1\leq k\leq m}|u(\gamma_k)|$ and all the points $\gamma_1,\dots,\gamma_m$ are poles of f in $\mathcal{D}_m$.

Suppose (α,β) is given such that (1) is valid. We define

(4) $$A_n(z)\cdot[B_n(z)]^{-1} \sim [d\cdot u(z)]^n \quad \text{as } n\to\infty,$$

If there is a function $\lambda(z)$ continuous in G, $\lambda(z) \neq 0$ for $z\in G$, such that

$$A_n(z)\cdot\left[B_n(z)\cdot d^n\cdot u^n(z)\right]^{-1} \to \lambda(z)$$

as $n\to\infty$ uniformly on each compact subset of G.

The main result of the present work is the following

THEOREM 4: Let $f\in\mathcal{H}(E)$ and the sequences (α,β) be given such that

$$A_n(z)\cdot\left[B_n(z)\right]^{-1} \sim \left[d\cdot u(z)\right]^n \quad \text{as } n\to\infty$$

in the sence of (4). If there exists a polynomial Q, $Q(z) = \prod_{k=1}^{m}(z-\gamma_k)$, $\gamma_1,\dots,\gamma_m\in G$ such that (3) holds for some q, $q<1$, then $\mathcal{D}_m \supseteq D_\sigma$ with

$\sigma = q^{-1}.\max_{1\le k\le m}|u(\gamma_k)|$.

It follows from Theorem 1 that all the poles of f in $\mathcal{D}_m$ can be situated at the points $\gamma_1,\dots,\gamma_m$.

&2

In this section we shall produce preliminary lemmas to be used in the proof of the theorem.

Lemma 1: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with $1 = \limsup_{n\to\infty}|a_n|^{1/n}$. Then there exists a monotone sequence of numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n/\lambda_{n+1} \to 1$ as $n\to\infty$ and $\limsup_{n\to\infty}|a_n.\lambda_n^{-1}| = 1$.

For each r, $r>0$ denote by K_r the disk $K_r=\{w, |w|<r\}$, $\partial K_r = C_r$ and for each pair (r_1,r_2), $r_1<r_2$ – by $D_{r_1r_2}$ the anulus $\{|w|, r_1<|w|<r_2\}$. Using lemma 1, we can present each function $\phi(w), \phi\in\mathcal{H}(K_r)$ in the following form: $\phi(w)=\sum_{n=0}^{\infty}\phi_n\lambda_n(w.r^{-1})^n$, where $\limsup_{n\to\infty}|\phi_n| = 1$ and $\{\lambda_n\}_{n=1}^{\infty}$ is the sequence indicated in Lemma 1.

Lemma 2: Let $\phi(w)=\sum_{n=0}^{\infty}\phi_n\lambda_n(w.r^{-1})^n$ be holomorphic in the disk K_r, $\limsup_{n\to\infty}|\phi_n| = 1$. Suppose, $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence of functions, holomorphic in $D_{r_1r_2}$, $r_1<r<r_2$ and let $(\phi\varphi_n)=\sum_{k=-\infty}^{\infty}\gamma_k^{(n)}(w.r^{-1})^k$. Then, if $\{\varphi_n\}$ converges as $n\to\infty$ uniformly on each compact subset of $D_{r_1r_2}$ to φ, $\varphi\ne 0$ in $D_{r_1r_2}$, the following estimates are valid:

$$0<\limsup_{n\to\infty}|\gamma_n^{(n)}.\lambda_n^{-1}|<\infty$$

Lemma 1 and Lemma 2 have been proved in [5].

The following lemma which is due to Gončar (not published) is the basis for our later considerations:

Lemma 3: Let $\phi(w)=\sum_{n=0}^{\infty}\phi_n\lambda_n(w.r^{-1})^n$, $\limsup_{n\to\infty}|\phi_n| = 1$. Let $\{\varphi_n(w)\}_{n=1}^{\infty}$ be functions, holomorphic in $D_{r_1r_2}$, containing C_r, such that $\varphi_n(w)\underset{n\to\infty}{\sim} w^n$ in $D_{r_1r_2}$ in the sence of (4). Then

$$\limsup_{n\to\infty}\left|\int_{C_r}\phi(w).\varphi_n^{-1}(w)dw\right|^{1/n} = r^{\pm 1}$$

(the integration on C_r is to be understood as an integration on each circle $C_{r-\varepsilon}$, $\varepsilon>0$, which is "very" close to C_r; $r_1<r-\varepsilon<r$)

Lemma 3 implies:

Lemma 4: Let the function $\phi(w)$ be holomorphic in the anulus D_{r_1r}.

Suppose, the functions $\{\varphi_n(w)\}_{n=1}^{\infty}$ are holomorphic in the anulus $D_{r_1 r_2}$, where $r_1 < r < r_2$ and $\varphi_n(w) \sim w^n$, $n \to \infty$, in the sence of (4) (in $D_{r_1 r_2}$). Then

$$(5) \qquad \limsup_{n\to\infty} \left| \int_{C_r} \Phi(w) . \varphi_n^{-1}(w)\, dw \right| = r^{-1}.$$

The proof of this lemma follows from Lemma 3. Indeed, in $D_{r_1 r_2}$ we have

$$\Phi(w) = \Phi_1(w) + \Phi_2(w)$$

where $\Phi_1 \in \mathcal{H}(\bar{C} - K_{r_1})$ and $\Phi_2 \in \mathcal{H}(K_r)$.

Lemma 3 implies

$$\limsup_{n\to\infty} \left| \int_{C_r} \Phi_2(w) \varphi_n^{-1}(w)\, dw \right|^{1/n} = r^{-1}$$

Then there exists a sequence Λ, $\Lambda \subset N$ such that

$$(6) \qquad \lim_{n\in\Lambda} \left| \int_{C_r} \Phi(w) \varphi_n^{-1}(w) dw - \int_{C_{r_1}} \Phi_1(w) . \varphi_n^{-1}(w) dw \right|^{1/n} = r^{-1}$$

On the other hand, for the function Φ_1 we have the estimates

$$\limsup_{n\to\infty} \left| \int_{C_{r_1}} \Phi_1(w) . \varphi_n(w)^{-1} dw \right|^{1/n} \leqq r_2^{-1}.$$

The last inequality and (6) yield that

$$\limsup_{n\in\Lambda} \left| \int_{C_r} \Phi(w)\, \varphi_n^{-1}(w)\, dw \right|^{1/n} \geqslant r^{-1}$$

But in the case being considered we have

$$\limsup_{n\to\infty} \left| \int_{C_r} \Phi(w) . \varphi_n^{-1}(w) dw \right|^{1/n} \leqq r^{-1}.$$

The last two estimates prove complitely Lemma 4.

Let us return to the case discussed in Theorem 4. Denote by $\varphi(w)$ the function which is inverse to the function $w(z) = d.u(z)$ in $D_{d.\varrho^{-1}, d}$. We have $\varphi' . w' = 1$. Since the domain G is double connected, the func - tion φ is holomorphic in $D_{d.\varrho^{-1}, d}$ and univalent.

Then Lemma 4 gives the following

Corollary: Let $\Psi(z)$ be a function, holomorphic in G_δ, $\varrho^{-1} < \delta < 1$. Then

$$\limsup_{n\to\infty} \left| \int_{\Gamma_\delta} \Psi(z) . B_{n-m}(z) \{A_{n+m+1}(z)\}^{-1} . u'(z) dz \right|^{1/n} = (d\delta)^{-1}$$

In this section we shall prove Theorem 4.

It follows from (2) and from the conditions of the theorem that

$$\int_E (P.f.B_{n-m})(z).\left[A_{n+m+1}(z)\right]^{-1}.Q_{n,m}(z)dz = 0 \tag{8}$$

for every polynomial P of degree not more than m-1, $P \not\equiv 0$. This gives

$$0 = \int_E (P.f.u'^{-1})(z).(Q_{n,m}.B_{n-m})(z).\left[A_{n+m+1}(z)\right]^{-1}(z).u'(z)dz \tag{9}$$

If $\nu(m)=m$, then it follows from Theorem 1 that all the poles of f are situated at the roots of the polynomial Q(z), i.e., at the points $\gamma_1,\dots,\gamma_m$. Consequently, $\gamma_k \in \mathcal{D}_m$ for k=1,...,m. Now it is easy to prove that $\mathcal{D}_m \supseteq D_\sigma$, where σ is given in the theorem.

Indeed, suppose that $|u(\gamma_1)| \le \dots \le |u(\gamma_m)|$ and let $\gamma_1,\dots,\gamma_{m1}$ be the poles of f in the domain $D_{|u(\gamma_m)|}$; $m_1 < m$. We set $F(z)=(f.Q)(z)$ and $F_1(z)=f(z).(z-\gamma_1)\dots(z-\gamma_{m_1})$.

Let $\varepsilon > 0$ be such a number that $f \in \mathcal{H}(D_{|u(\gamma_m)|} - D_{|u(\gamma_m)|-\varepsilon})$. Since $F_1 \in \mathcal{H}(D_{|u(\gamma_m)|})$ it follows from (3) and from (8) that

$$\limsup_{n\to\infty}\left|\int_{\Gamma_{u(\gamma_m)-\varepsilon}} F_1(z).\left(Q.B_{n-m}\right)(z).\left[A_{n+m+1}(z)\right]^{-1} dz\right|^{1/n} < q.\left(d\left(|u(\gamma_m)|-\varepsilon\right)\right)^{-1}$$

This means, because of the arbitrariness of ε, that $(F_1Q).u'^{-1} \in \mathcal{H}(G_{|u(\gamma_m)|.q^{-1}})$. But $F_1Q(z) \equiv F(z).\prod_{k=1}^{m_1}(z-\gamma_k)$. Consequently, $(F.u'^{-1}) \in \mathcal{H}(G_{q^{-1}.|u(\gamma_m)|})$. Since f is holomorphic on E and $u'^{-1}(z)$ - in G, the function F is holomorphic in the whole domain $D_{|u(\gamma_m)|.q^{-1}}$. But $F \in \mathcal{H}(\mathcal{D}_m)$. Consequently, in this case, when $\nu(m)=m$, the domain $\mathcal{D}_m$ contains D_σ, where σ is given by $\sigma = q^{-1}.(\max|u(\gamma_k)|, k=1,\dots,m)$.

Suppose, now, that $\nu(m)<m$. Let $\gamma_{k_1},\dots,\gamma_{k_\mu}$ be the poles of f in $\mathcal{D}_m$; $\mu < m$. We set $F_1(z)=f(z).\prod_{i=1}^{\mu}(z-\gamma_{k_i})$. We shall prove that in the conditions of the theorem the function F_1 has a holomorphic continuation in the whole domain D; this will mean, that $\mathcal{D}_m = D$.

We set $\delta_0 = \sup\{\delta, F_1 \in \mathcal{H}(D_\delta)\}$. Obviously, $\delta_0 = \sup\{\delta, F_1.u'^{-1} \in \mathcal{H}(G_\delta)\} = \sup\{\delta, F.u'^{-1} \in \mathcal{H}(G_\delta)\} \ge \varrho^{-1}(G)$.

Suppose that $\delta_0 < 1$. If $Q(z) \neq 0$ for $z \in \Gamma_{\delta_0}$, then

$$\left((Q_{n,m}\cdot B_{n-m})^{-1}\cdot A_{n+m+1}\right)(\varphi(w)) \sim w^n, \; n\to\infty$$

in the sence of (4) and Lemma 4 (Corollary) is applicable. This implies

$$\limsup_{n\to\infty}\left|\int_{\Gamma\delta_0}(F_1\cdot Q_{n,m}\cdot B_{n-m})(z)\cdot\left(A_{n+m+1}(z)\right)^{-1}dz\right|^{1/n} = (d\delta_0)^{-1}$$

But the last equality contradicts (8). Consequently, if $\delta_0 < 1$, then there is a zero γ_1 of the polynomial $Q(z)$ such that $\gamma_1 \in \Gamma\delta_0$, $1\le l\le m$. Then Lemma 4 implies that

$$\limsup_{n\to\infty}\left|\int_{\Gamma\delta_0}(F_1\cdot Q\cdot B_{n-m})(z)\cdot(A_{n-m-1}(z))^{-1}dz\right|^{1/n} = (d\delta_0)^{-1}$$

This is a contradiction to the estimate

$$\limsup_{n\to\infty}\left|\int_{\Gamma\delta_0}(F_1\cdot Q\cdot B_{n-m})(z)\cdot\left(A_{n-m-1}(z)\right)^{-1}dz\right|^{1/n} \leqq q(d\delta_0)^{-1}$$

which follows from (8) and from the conditions of the theorem (remember, that $q<1$). Thus $\delta_0=1$ and $\mathcal{D}_m = D$.

We have proved that in the conditions of Theorem 4 either the function f has exactly m poles in $\mathcal{D}_m$ and $\mathcal{D}_m = D_\sigma$ with $\sigma = q^{-1}\cdot\max_{1\le k\le m}|u(\ _k)|$, or $\mathcal{D}_m = D$, if the numbers of the poles of f in $\mathcal{D}_m$ is less than m. This completes the proof.

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