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SOME GENERALIZATION OF LEBESGUE AND SOBOLEV SPACES

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The theory of Lebesgue spaces L_p for p=const is very known.Basic information see e.g. in [1] and [2] . An effort about of generalization of L_p given by so-called Orlicz spaces, which are study e.g. in[2] and [3]. Other way of generalization shown by I.I.Sharapudinov in [4]. In this paper we give a completation of [4] and generalization of Sobolev spaces, that for p=const are described e.g. in [2] .

Let $I = [a,b]$ be an closed interval in $\mathbb{R}$. Let $p(t)$ be a real mesurable function defined on I, for which $p(t) \geq 1$. Next we assume that for the value $p_o = \operatorname{supess} p(t) < \infty$ is true. Then we shall study the set of all real mesurable functions f(t) defined on I for which

$$\int_I |f(t)|^{p(t)}\, dt < \infty$$

holds. Further we shall identify two functions f_1 and f_2 is $f_1(t) = f_2(t)$ a.e. on I. Then we obtain the set of functions denoted by $L^{p(t)}$. Here we define

$$|||f|||_p = \{ \int_I |f(t)|^{p(t)}\, dt\}^{1/p_o} \tag{1}$$

Distributional derivatives $f^{(i)}(t)$ of order i of real function f(t) defined and mesurable on I are introducet so that for every function $\psi(t)$ from $C_o^\infty(I)$ and for $i=0,1,2,...$

$$\int_I f^{(i)}(t).\psi(t)dt=(-1)^i.\int_I f(t)\ .\psi^{(i)}(t)dt$$

holds. Then for nonnegative integer k we denote by $W^{k,p(t)}$ the set of all real mesurable function f(t) defined on I, for which

$$\sum_{i=0}^{k} \int_I |f^{(i)}(t)|^{p(t)}\, dt < \infty$$

Here we define

$$|||f|||_{k,p} = \sum_{i=0}^{k} |||f^{(i)}|||_p \tag{2}$$

Formula (1) [(2) resp.] define the deminorm on $L^{p(t)}$ [$W^{k,p(t)}$ resp.], i.e. it is value for which

1. $|||f||| \geq 0$ ($|||f||| = 0 \iff f(t)=0$ a.e.),

2. $|||f + g||| \leq |||f||| + |||g|||$

holds, but

3. $|||c.f||| \leq \text{supess}|c|^{p(t)/p_o} . |||f|||$

(see in [4]). We introduce the metric in $L^{p(t)}$ [$W^{k,p(t)}$ resp.] as follows $\rho(f,g) = |||f-g|||_p$ [$\rho_k(f,g) = |||f-g|||_{k,p}$ resp.]

We emphasize some results from [4] .

THEOREM 1. For linearity of topological space $L^{p(t)}$ will condition $p_o<\infty$ be necessary and sufficient.

Hence it follows necessity of condition for p(t) in introduction. The charakterization of deminorm gives

LEMMA 2. Let $f,g \in L^{p(t)}$ and c=const. Then a, $|||f+g|||_p \leq |||f|||_p + |||g|||_p$ b, $|||c.f|||_p \leq \text{supess } |c|^{p(t)/p_o} . |||f|||_p$.

COROLLARY 3. Let $f \in L^{p(t)}$ and c=const. Then $|||c.f|||_p \leq \max\{1;|c|\} \times \times |||f|||_p$.

For definition of the norm in $L^{p(t)}$ is important a local convexity. There holds

PROPOSITION 4. $L^{p(t)}$ is a convex set in sense, that from $\int_I |f_j(t)|^{p(t)} dt \leq \delta$ for j=1,2 and for real $\lambda \in [0;1]$ there follows

$$\int_I |\lambda f_1(t) + (1-\lambda) f_2(t)|^{p(t)} dt \leq \lambda \int_I |f_1(t)|^{p(t)} dt + (1-\lambda) \int_I |f_2(t)|^{p(t)} dt \leq \delta .$$

THEOREM 5. $L^{p(t)}$ is a normable linear space. One of equivalent norms is defined by $\|f\|_p = \inf\{\alpha>0: \int_I |f(t)/\alpha|^{p(t)} dt \leq 1 \}$.

We define the convergence in $L^{p(t)}$ as follows: The sequence $\{f_n\}$ from $L^{p(t)}$ is said to be convergent to f in $L^{p(t)}$ if for any $\varepsilon > 0$ there exists a natural n_o for which $\forall n \geq n_o : \int_I |f_n(t)-f(t)|^{p(t)} dt<\varepsilon$ holds.

LEMMA 6. Operations of addition and multiplication by real const. in $L^{p(t)}$ are continuous in metric ρ .

In paper [4] there is proved the reflexivity of spaces $L^{p(t)}$ for case

$$1 < \text{infess } p(t) \leq p_o < \infty \tag{3}$$

using the general form of continuous linear functional on $L^{p(t)}$ and using a duality. Hence from (3) and from

$$1/p(t) + 1/q(t) = 1 \quad \text{a.e. on } I \tag{4}$$

we have $[L^{p(t)}]^* = L^{q(t)}$. also there holds the general Hölder inequality:

THEOREM 7. Let (3) and (4) be true. If $f \in L^{p(t)}$ and $g \in L^{q(t)}$, then

$$\int_I |f(t).g(t)|dt \le [1/\text{infess } p(t) + 1/\text{infess } q(t)].\text{supess}\{\int_I |f(t)|^{p(t)}dt\}^{1/p(t)}.\text{supess}\{\int_I |g(t)|^{q(t)}dt\}^{1/q(t)},$$

or

$$\int_I |f(t).g(t)|dt \le r.\|f\|_p.\|g\|_q, \quad (r \le 1/\text{infess } p(t) + 1/\text{infess } q(t))$$

respectively.

Hence $f.g \in L^1$ follows.

THEOREM 8. $L^{p(t)}$ is a complete metric space.

P r o o f: Let $\{f_n(t)\}$ be fundamental sequence in $L^{p(t)}$, i.e. $|||f_k - f_l|||_p \to 0$ for $k,l \to \infty$. We choose subsequence $\{f_{n_i}\}$ so that $|||f_{n_{i+1}} - f_{n_i}|||_p \le 2^{-i}$. Denote

$$g_k(t) = \sum_{i=1}^{k} |f_{n_{i+1}}(t) - f_{n_i}(t)|$$

and

$$g(t) = \lim_{k\to\infty} g_k(t).$$

Then from Lemma 2 for every natural k $|||g_k|||_p \le 1$ follow and from Fatou´s lemma we have that $|||g|||_p \le 1$. Because $p_o < \infty$ then $g(t) < \infty$ a.e. on I and the series

$$f_{n_1}(t) + \sum_{i=1}^{\infty} \{f_{n_{i+1}}(t) - f_{n_i}(t)\} = F(t) \qquad (5)$$

is absolutely convergent a.e. on I. We define the function

$$f(t) = \begin{cases} F(t) & \text{for such } t \in I \text{ where (5) converges,} \\ 0 & \text{for other } t \text{ (on the set with mesure 0).} \end{cases}$$

Because $f_{n_k} = f_{n_1} + \sum_{i=1}^{k-1} \{f_{n_{i+1}} - f_{n_i}\}$, then

$$f(t) = \lim_{k\to\infty} f_{n_k}(t) \quad \text{a.e. on } I .$$

Let us prove that $f(t) = \lim_{n\to\infty} f_n(t)$ in $L^{p(t)}$. We choose $\varepsilon > 0$. Then there exists a natural n_o, such that $\forall\, m,n \ge n_o : |||f_n - f_m|||_p < \varepsilon$ is true. From Fatou´s lemma for every $m \ge n_o$

$$\int_I |f(t) - f_m(t)|^{p(t)}dt \le \liminf_{i\to\infty} \int_I |f_{n_i}(t) - f_m(t)|^{p(t)}dt \le \varepsilon^{p_o}, \qquad (6)$$

i.e. $f - f_m \in L^{p(t)}$ follows. Because $f(t) = \{f(t) - f_m(t)\} + f_m(t)$, then $f \in L^{p(t)}$, at the same time from (6) we obtain, that $|||f - f_m|||_p \to 0$ for $m \to \infty$.

QED.

From Theorem 7 for $p(t) \ge 1$ and $p_o < \infty$ there follows $L^{p(t)} \subset L^1$.

THEOREM 9. The space C_o^∞ is dense in $L^{p(t)}$.

P r o o f : Because every continuous function lies in $L^{p(t)}$, $C_o^\infty \subset L^{p(t)} \subset L^1$ and C_o^∞ is dense in L^1, then for $p(t) \ge 1$ and $p_o < \infty$

C_o^∞ is dense in $L^{p(t)}$. QED.

THEOREM 10. The space of bounded mesurable functions is dense in $L^{p(t)}$.

P r o o f : We will show that for any function $f \in L^{p(t)}$ there exists a sequence of bounded functions $\{g_n\}$ such that for every $\varepsilon>0$ there exists a natural n_o for which $\forall n \geq n_o$: $\int_I |f(t)-g_n(t)|^{p(t)} dt < \varepsilon$. We can see that the bounded function $g(t)$ lies in $L^{p(t)}$ since $|g(t)|^{p(t)}$ is integrable. Really, there exists a real $K \geq 0$ such that $|g(t)| \leq K$. Then $\int_I |g(t)|^{p(t)} dt \leq \operatorname{supess} K^{p(t)}.(b-a) < \infty$. Denote

$$g_n(t) = \begin{cases} f(t) & \text{for such } t, \text{for which } |f(t)| \leq n, \\ n & \text{for other } t. \end{cases}$$

Then for $t \in I$ $\lim_{n\to\infty} g_n(t) = f(t)$, i.e. $\lim_{n\to\infty} |f(t)-g_n(t)| = 0$. Hence using Fatou´s lemma $\|f-g_n\|_1 \to 0$ for $n \to \infty$ follows. From Theorem 7 we have $|||f-g_n|||_p \to 0$ for $n \to \infty$. QED.

THEOREM 11. The family of all functions with finite set of values is dense in $L^{p(t)}$.

P r o o f follows from Theorem 9, because the family of all functions with finite set of values is dense in C_o^∞. QED.

THEOREM 12. The set of polynomials on I is dense in $L^{p(t)}$.

P r o o f follows from Weierstrass´ theorem for uniform approximation of continuous function by polynomials on compact and from Theorem 9. QED.

THEOREM 13. $L^{p(t)}$ is separable.

P r o o f follows from Theorem 12 and from the fact that every polynomial can be on compact uniformly approximated by polynomials with rational coefficients. QED.

THEOREM 14. Let $1 \leq p(t) < q(t)$ for every $t \in I$ and $q_o < \infty$. Then $L^{q(t)} \subset L^{p(t)}$.

P r o o f: Let $f \in L^{p(t)}$. Denote $r(t) = q(t)/p(t) > 1$ and $s(t) = r(t)/[r(t)-1]$. From Theorem 7 we get

$$\int_I |f(t)|^{p(t)} dt \leq \operatorname{supess}\{\int_I |f(t)|^{p(t).r(t)} dt\}^{1/r(t)} . \operatorname{supess}\{\int_I dt\}^{1/s(t)} \times$$

$$\times [1/\operatorname{infess} r(t) + 1/\operatorname{infess} s(t)] \leq$$

$$\leq 2.\operatorname{supess} (b-a)^{[q(t)-p(t)]/q(t)} . \operatorname{supess}\{\int_I |f(t)|^{q(t)} dt\}^{p(t)/q(t)},$$

Which is finite with respect to $0 < p(t)/q(t) < 1$ and $0 < [q(t)-p(t)]/q(t) < 1$. QED.

Denote lin V the linear span of set V.

THEOREM 15. Let $r(t) \geq 1, r_o < \infty, s(t) \geq 1$ and $s_o < \infty$. If $p(t) = \min\{r(t), s(t)\}$ and $q(t) = \max\{r(t), s(t)\}$ for every $t \in I$, then

1, $L^{q(t)} = L^{r(t)} \cap L^{s(t)}$ 2, $L^{p(t)} = \operatorname{lin}\{L^{r(t)} \cup L^{s(t)}\}$.

P r o o f: 1, From Theorem 14 $L^{q(t)} \subset L^{r(t)} \cap L^{s(t)}$ follow. We also have $\int_I |f(t)|^{q(t)} dt \le \max\{\int_I |f(t)|^{r(t)} dt; \int_I |f(t)|^{s(t)} dt\}$, i.e. $L^{r(t)} \cap L^{s(t)} \subset L^{q(t)}$. Therefore $L^{q(t)} = L^{r(t)} \cap L^{s(t)}$.

2, From Theorem 14 we have $\operatorname{lin}\{L^{r(t)} \cup L^{s(t)}\} \subset L^{p(t)}$. Denote $E_1 = \{ t \in I : r(t) \ge s(t)\}$ and $E_2 = \{ t \in I : r(t) < s(t)\}$.

For $f \in L^{p(t)}$ we define

$$f_1(t) = \begin{cases} f(t) & \text{for } t \in E_1, \\ 0 & \text{for } t \in E_2 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0 & \text{for } t \in E_1, \\ f(t) & \text{for } t \in E_2 . \end{cases}$$

We can see that $f_1 \in L^{r(t)}$ and $f_2 \in L^{s(t)}$, i.e. $f = f_1 + f_2$ lie in $\operatorname{lin}\{L^{r(t)} \cup L^{s(t)}\}$. So $L^{p(t)} \subset \operatorname{lin}\{L^{r(t)} \cup L^{s(t)}\}$. Therefore

$$L^{p(t)} = \operatorname{lin}\{ L^{r(t)} \cup L^{s(t)} \} .$$ QED.

LEMMA 16. If $f, g \in W^{k,p(t)}$ and $c = \text{const}$, then

a, $|||f + g|||_{k,p} \le |||f|||_{k,p} + |||g|||_{k,p}$,

b, $|||c.f|||_{k,p} \le \operatorname{supess} |c|^{p(t)/p_o} . |||f|||_{k,p}$.

P r o o f: We can prove this Lemma similarly as Lemma 2 (Lemma in [4]) with respect to $f^{(i)}, g^{(i)} \in L^{p(t)}$ for $i = 0, 1, \dots, k$.

THEOREM 17. The condition $p_o < \infty$ is necessary and sufficient for linearity of $W^{k,p(t)}$.

P r o o f : S u f f i c i e n c y we prove easily by using Lemma 16.

N e c e s s i t y. Let $p_o = \infty$. Denote $E_n = \{t: n-1 \le p(t) < n\}$ for $n = 1, 2, \dots$. Then we can construct the function $g(t)$ continuous for such positive integer k, for which $g^{(k)}(t) = c_n > 0$ if $t \in E_n$, where $\int_{E_n} c_n^{p(t)} dt = 1/n^2$ for c_n. We have

$$\int_I |g^{(k)}(t)|^{p(t)} dt = \sum_{n=1}^{\infty} \int_{E_n} c_n^{p(t)} dt = \sum_{n=1}^{\infty} 1/n^2 < \infty .$$

Since $p(t) \ge 1$ and $g^{(k)} \in L^{p(t)}$, then $g^{(k)} \in L^1$ and hence $G(t) = \int_a^t g^{(k)}(x)\, dx$. $G(t)$ is not only continuous, but also absolutely continuous. Therefore $G(t)$ has derivative (at $]a;b[$), equal to $g^{(k)}(t)$ almost everywhere. Then we can denote $G(t) = g^{(k-1)}(t)$, where $G(t) \in L^{p(t)}$. We can proceed from k into 0 . Hence

$$|||g|||_{k,p} = \sum_{i=0}^{k} |||g^{(i)}|||_p < \infty .$$

Now we take $2g(t)$. Then we have

$$\int_I |2.g^{(k)}(t)|^{p(t)} dt = \sum_{n=1}^{\infty} \int_{E_n} |2g^{(k)}(t)|^{p(t)} dt \le \sum_{n=1}^{\infty} 2^n . \int_{E_n} c_n^{p(t)} dt = \sum_{n=1}^{\infty} 2^n/n^2$$

It is divergent, therefore $W^{k,p(t)}$ is not linear for $p_o = \infty$. QED.

THEOREM 18. $W^{k,p(t)}$ is a normable linear space. One of the equivalent norms is defined by $\|f\|_{k,p} = \sum_{i=0}^{k} \inf\{\alpha_i > 0: \int_I |f^{(i)}(t)/\alpha_i|^{p(t)} dt \le 1\}$.

P r o o f: We can step by using standard methods.

THEOREM 19. $W^{k,p(t)}$ is complete metric space.

P r o o f follows from completeness of $L^{p(t)}$.

THEOREM 20. Let k be a positive integer. Then $W^{k,p(t)} \subset L^{p(t)}$ and $W^{0,p(t)} = L^{p(t)}$.

P r o o f implies from the definition of the space $W^{k,p(t)}$.

THEOREM 21. Let for integers $0 \le k_1 \le k_2$. Then $W^{k_1,p(t)} \subset W^{k_2,p(t)}$.

P r o o f: Since $0 \le k_1 \le k_2$, then for $f \in W^{k_2,p(t)}$ we have $|||f^{(i)}|||_p$ is finite for every $i=0,1,\dots,k_2$. Therefore even for $i=0,1,\dots,k_1$ we have $|||f|||_{k_1,p} \le |||f|||_{k_2,p} < \infty$. QED.

THEOREM 22. Let $1 \le p(t) < q(t)$ for any $t \in I$ and $q_o < \infty$. Let k be a nonnegative integer. Then $W^{k,q(t)} \subset W^{k,p(t)}$.

P r o o f follows from Theorem 14.

THEOREM 23. Let k_1, k_2 be nonnegative integers for which $0 \le k_1 \le k_2$. Let $1 \le p(t) < q(t)$ for any $t \in I$ and $q_o < \infty$. Then $W^{k_2,q(t)} \subset W^{k_1,p(t)}$.

P r o o f follows from Theorems 21 and 22.

THEOREM 24. $W^{k,p(t)}$ is separable.

P r o o f follows from Theorems 12 and 13.

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