КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 493—499

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.493-499

A NOTE ON THE GENERAL MULTIVARIATE MOMENT PROBLEM

Marek A. Kowalski

1. Introduction and summary. This paper aims to generalize the following theorem.

Theorem 1. For any sequence $\{\mu_{j_1 j_2 \dots j_n}\}$ of complex numbers there exist infinitely many complex Radon measures μ on $\mathbb{R}^n$ such that

$$(1)\qquad \int_{\mathbb{R}^n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} d\mu(x_1, x_2, \dots, x_n) = \mu_{j_1 j_2 \dots j_n}, \quad j_1, \dots, j_n = 0, 1, \dots$$

For n=1 this theorem essentially belongs to Borel, see [2, pp. 74-75] or [7]. The extension of his result to dimension $n>1$ can be found in [1].

In this paper we consider μ in (1) of the form

$$(2)\qquad d\mu = \varrho\, d\phi^{(n)}$$

where $d\phi^{(n)}(x_1, x_2, \dots, x_n) = d\phi(x_1) d\phi(x_2) \dots d\phi(x_n)$, $\varrho \in L_2(\mathbb{R}^n, \phi^{(n)})$ and ϕ is a one dimensional positive Radon measure. First we prove that for some measure ϕ which is absolutely continuous with respect to the Borel measure, (1) holds for infinitely many functions ϱ. Next we prove that for some atom measure ϕ, (1) holds also for infinitely functions ϱ. We also show that there exists an atom measure ϕ such that (1) holds exactly for one function ϱ. These results seem to be new even for n=1.

This paper is based on the proof technique used in [4] and [5] where the orthogonality of polynomials in several variables is considered.

2. Notation. Let π_n^∞ be the vector space of all polynomials in n real variables with complex coefficients. By π_n^k we mean the subspace of π_n^∞ consisting of polynomials of degree at most k. Let $\{P_i^k\}_{k=0}^{\infty}{}_{i=1}^{r_n^k}$ be a basis of π_n^∞. The upper index k denotes the degree of P_i^k and $r_n^k = \binom{n+k-1}{k}$ is the total number of linearly independent polynomials from π_n^k with respect to π_n^{k-1} (i.e., $t_1 P_1^k + t_2 P_2^k + \ldots + t_{r_n^k} P_{r_n^k}^k \in \pi_n^{k-1}$ implies that $\sum_i |t_i| = 0$). Let

$$\vec{P}_k := [P_1^k, P_2^k, \ldots, P_{r_n^k}^k]^T, \quad k=0,1,\ldots ,$$

and

$$p_1 := P_1^0,\ p_2 := P_1^1,\ p_3 := P_2^1, \ldots, p_{i+r_n^k} := P_i^k, \quad k=0,1,\ldots\ ;\ i=1,\ldots,r_n^k .$$

For convenience we use the notation $\{\vec{P}_k\}_{k=0}^\infty$ or $\{p_i\}_{i=1}^\infty$ instead of $\{P_i\}_{k=0}^{\infty}{}_{i=1}^{r_n^k}$. Let $\vec{W} = [w_1, w_2, \ldots, w_l]^T$ where $w_1, w_2, \ldots, w_l$ are polynomials in $x_1, x_2, \ldots, x_n$. We define

$$\vec{xW} := [x_1\vec{W}^T \,|\, x_2\vec{W}^T \,|\ldots|\, x_n\vec{W}^T]^T.$$

For instance if n=2 and $\vec{W} = [x_1^2, x_1x_2, x_2^2]^T$ then

$$\vec{xW} = [x_1^3, x_1^2x_2, x_1x_2^2, x_2x_1^2, x_1x_2^2, x_2^3]^T.$$

3. Main results. Consider two bases $\{\vec{P}_k\}_{k=0}^\infty = \{p_i\}_{i=1}^\infty$ and $\{\vec{Q}_k\}_{k=0}^\infty = \{q_i\}_{i=1}^\infty$ in the space π_n^∞. A polynomial q_i can be written in the form $q_i = \sum_{l=1}^\infty c_{il} p_l$, where $c_{il} \in \mathbb{C}$ vanishes for sufficiently large l. Assume that $\{q_i\}_{i=1}^\infty$ is an orthonormal sequence in the space $L_2(\mathbb{R}^n,\mu)$. Let j be a given integer. The proof technique of Theorem 2 from [5] allows us to get the following theorem.

Theorem A. There exists a function $\varrho \in L_2(\mathbb{R}^n,\mu)$ satisfying

$$\int_{\mathbb{R}^n} p_i \varrho \, d\mu = \delta_{ij} \quad , \quad i=1,2,\ldots ,$$

(where δ_{ij} is the Kronecker delta) iff the series $\sum_{i=1}^\infty |c_{ij}|^2$ is convergent.

Sketch of the proof. Consider the linear functional l on

π_n^∞ defined by

(3) $$\mathbf{l}(p_i) = \delta_{ij}\ ,\ i=1,2,\dots\ .$$

Let $w = \sum_{k=1}^{n} a_k q_k$, $a_k \in \mathbb{C}$, be a given polynomial. Then

$$\mathbf{l}(w) = \sum_{k\leqslant n} a_k \mathbf{l}(q_k) = \sum_{k=1}^{n} a_k \mathbf{l}(\sum_{i=1}^{\infty} c_{ki}\, p_i) = \sum_{k=1}^{n} a_k c_{kj}\ .$$

Applying Schwarz' inequality we get

$$|\mathbf{l}(w)| = |\sum_{k=1}^{n} a_k c_{kj}| \leqslant (\sum_{k=1}^{n} |c_{kj}|^2)^{1/2} (\sum_{k=1}^{n} |a_k|^2)^{1/2} = (\sum_{k=1}^{n} |c_{kj}|^2)^{1/2} \|w\|_2$$

where $\|w\|_2 = (\int_{R^n} |w|^2 d\mu)^{1/2}$. Since this estimate is sharp, we get

$$\|\mathbf{l}\| = \sup\{|\mathbf{l}(w)| : w \in \pi_n^\infty,\ \|w\|_2 = 1\} = (\sum_{k=1}^{\infty} |c_{kj}|^2)^{1/2}\ .$$

This proves that (3) defines the continuous functional with respect to $\|.\|$ iff the series $\sum_{k=1}^{\infty} |c_{kj}|^2$ is convergent. To get the desired result it is enough to apply the Hahn-Banach theorem and the Riesz theorem in the same way as in the proof of the comparison criterion of orthogonality from [5]. ■

Furthermore Theorem 3 from [5] can be restated in a more general form. Namely we have

<u>Theorem B.</u> There exist a positive Radon measure ϕ on $\mathbb{R}$ and real polynomials $\{R_i^k\}_{k=0}^{\infty}{}_{i=1}^{r_n^k}$ such that

(i) $$\int_{R^n} R_i^k R_j^l \, d\phi^{(n)} = \delta_{kl}\,\delta_{ij}\ .$$

(ii) for $r_n^i x r_n^k$ matrices G_{ik} defined by

(4) $$\vec{R}_i = \sum_{k=0}^{i} G_{ik} \vec{P}_k,$$

we have $\| G_{ij} \|_\infty < 2^{-i+1} |G_{00}|$, $i=1,2,\dots$; $j=0,1,\dots$. ($\|.\|_\infty$ denotes the infinite matrix norm.)

<u>Sketch of the proof.</u> We shall find the polynomials R_i^k in such a way that the corresponding recursion formula takes the form

(5) $$\vec{R}_{k+1} = d_k \overrightarrow{xR}_k + f_k \vec{R}_{k-1},\quad k=0,1,\dots\ ;\ f_0 = 0,\ Q_{-1} = 0.$$

Let the matrices $A_j^{(k)}$, G_{ij} be defined by the equations

$$\overrightarrow{xP}_k = \sum_{j=0}^{k+1} A_j^{(k)} \vec{P}_j\ ,\quad \vec{R}_k = \sum_{j=0}^{k} G_{kj}\, \vec{P}_j\ ,\quad k=0,1,\dots\ .$$

Due to (5), proceeding similarly as in the proof of Theorem 3 from [5], we get

(6) $$G_{k+1,j} = d_k (\sum_{i=j-1}^{k} [G_{ki}]\, A_j^{(i)}) + f_k G_{k-1,j}\ ,\quad k=0,1,\dots;\ j=0,\dots,k+1,$$

where G_{st} for $s<0$ or $t>s$ are zero matrices and $[M]$ denotes the n-block matrix of the form

$$[M] = \begin{bmatrix} M & & & 0 \\ & M & & \\ & & \ddots & \\ 0 & & & M \end{bmatrix} .$$

Then equation (6) yields

$$(7) \qquad \max_{0\leqslant l\leqslant k+1} \|G_{k+1\,l}\| \leqslant (\|d_k\|\Delta_k + \|f_k\|) \max_{\substack{s=k,k-1\\ 0\leqslant t\leqslant s}} \|G_{st}\|$$

where $\Delta_k = (k+1) \max_{\substack{0\leqslant l\leqslant k+1\\ 0\leqslant i\leqslant k}} \|A_l^{(i)}\|$ and $\|\cdot\| = \|\cdot\|_\infty$.

Let $S_p = \{ i\geqslant 0 : \lceil \frac{i+1}{n} \rceil = p \}$ ($\lceil y \rceil$ = the smalest integer $\geqslant y$). We now choose positive numbers α_j, γ_j such that $\alpha_1 \leqslant \min_{i\in S_1} 1/(8\Delta_i)$, $\alpha_j \leqslant \min_{i\in S_j} \alpha_{j-1}/[8(1+\Delta_i\alpha_{j-1})]$ and $\gamma_j = \alpha_j/\alpha_{j-1}$ $(j=2,3,\dots)$. So we have

$$(8) \qquad \begin{cases} \alpha_1 > 0,\ 0 < \alpha_{i+1} < \alpha_i/8, \quad \gamma_i < 1/8, \quad i=1,2,\dots , \\ \Delta_k \alpha_{\lceil \frac{k+1}{n} \rceil} + \gamma_{\lceil \frac{k+1}{n} \rceil} \leqslant 1/8, \quad k=0,1,\dots;\ \gamma_1 = 0. \end{cases}$$

Consider polynomials in one variable defined by

$$(9) \qquad r_{l+1}(y) = \alpha_{l+1} y r_l(y) - \gamma_{l+1} r_{l-1}(y),\ l=0,1,\dots ;\ r_{-1}=0,\ r_0=1.$$

Due to the Favard theorem there exists a positive Radon measure ϕ on $\mathbb{R}$ such that $\int_{\mathbb{R}} r_k(y) r_l(y)\, d\phi(y) = \delta_{kl}$ for $k,l=0,1,\dots$. Thus, for the measure ϕ and real polynomials R_i^k,

$$\{R_i^k\}_{k=0}^{\infty}{}_{i=1}^{r_n^k} = \{ r_{j_1}(x_1) r_{j_2}(x_2) \dots r_{j_n}(x_n) \}_{k=0}^{\infty}{}_{j_1+j_2+\dots+j_n=k},$$

the equation (1) holds. From (9) it follows that matrices d_k, f_k in the formula (5) can be chosen in such a way that they have exactly one nonvanishing element in each row. These nonzero elements are from the sets $\{\alpha_{k+1}, \alpha_k, \dots, \alpha_{\lceil (k+1)/n \rceil}\}$ and $\{-\gamma_{k+1}, -\gamma_k, \dots, -\gamma_{\lceil (k+1)/n \rceil}\}$ respectively. Hence $\Delta_k\|d_k\| + \|f_k\| < 1/4$. Then (7) yields

$$\max_{0\leqslant l\leqslant k} \|G_{kl}\| < 2^{-k+1} |G_{00}| , \quad k=1,2,\dots ,$$

as claimed. ■

Corollary 1. There exist a positive Radon measure ϕ on $\mathbb{R}$ and a function ϱ from $L_2(\mathbb{R}^n,\phi^{(n)})$ such that

$$\int_{\mathbb{R}^n} p_k \varrho \, d\phi^{(n)} = \delta_{kj}, \quad k=1,2,\ldots .$$

Proof. Consider a measure ϕ and polynomials $\{\vec{R}_k\}_{k=0}^{\infty} = \{r_i\}_{i=1}^{\infty}$ given by Theorem B. Due to Theorem A it is enough to verify the condition $\sum_{i=1}^{\infty} |e_{ij}|^2 < \infty$ where

$$(4') \qquad r_i = \sum_{l=1}^{\infty} e_{il} p_l, \quad i=1,2,\ldots .$$

Comparing (4′) and (4) we conclude that $\sum_{i=1}^{\infty} |e_{ij}| = \sum_{i=0}^{\infty} \|(G_{is})_t\|_2^2$ where s and t are chosen such that $p_j = P_t^s$ and $(G_{is})_t$ is the t-th column of G_{is}. Then

$$\sum_{i=0}^{\infty} \|(G_{is})_t\|_2^2 \leqslant \sum_{i=0}^{\infty} r_n^i \|G_{is}\|_{\infty}^2 < \sum_{i=0}^{\infty} \binom{n+i-1}{i} 2^{-2(i-1)} |G_{00}|^2 < |G_{00}|^2 2^{n+2} < \infty,$$

which completes the proof. ■

Let $\{m_i\}_{i=1}^{\infty}$ be an arbitrary basis of the space Π_n^{∞}. Assume without loss of generality that $\deg m_i \leqslant \deg m_{i+1} \leqslant 1 + \deg m_i$. We are now in a position to prove the following theorem.

Theorem C. For any sequence $S = \{\mu_i\}_{i=1}^{\infty}$ of complex numbers there exist a positive Radon measure ϕ on $\mathbb{R}$ and a function $\varrho \in L_2(\mathbb{R}^n,\phi^{(n)})$ such that

$$(10) \qquad \int_{\mathbb{R}^n} m_i \varrho \, d\phi^{(n)} = \mu_i, \quad i=1,2,\ldots .$$

Moreover the measure ϕ can be chosen in such a way that either

(i) ϕ is absolutely continuous with respect to the Borel measure and (10) holds for infinitely many functions ϱ, or

(ii) ϕ is an atom measure and (10) holds for a unique function ϱ or ϕ is atom and (10) is satisfied for infinitely many functions ϱ.

Proof. If S is the zero sequence then (10) is valid for $\phi = 0$. Assume thus that $S \neq 0$. Let j be the index of the first nonvanishing coefficient of S. Define

$$(11) \qquad p_k = m_k \text{ if } k<j, \quad p_k = m_j/\mu_j \text{ if } k=j, \quad p_k = m_k - \mu_k m_j/\mu_j \text{ if } k>j.$$

By Corollary 1 there exist a measure ϕ and a function $\varrho \in L_2(\mathbb{R}^n,\phi^{(n)})$ such that

$$\int_{R^n} p_k \varrho \, d\phi^{(n)} = \delta_{kj} \, , \; k=1,2,\ldots \, .$$

This and (11) immediately gives (10).

From the proof of Theorem B it follows that the measure ϕ in (10) can be chosen as any solution of the Hamburger moment problem associated with orthonormal polynomials r_l satisfying the recursion formula (9). (For simplicity we shall call this problem the H(r) moment problem). By (8) and (9) we get

(12) $$|r_{l+1}(0)| < |r_{l-1}(0)|/8, \; l=1,2,\ldots$$

which implies the convergence of the series $\sum_{l=0}^{\infty} r_l(0)^2$ and $\sum_{l=0}^{\infty} h_l(0)^2$ where $h_i(x) = \int_R (r_i(x)-r_i(t))/(x-t) \, d\phi(t)$. Due to [8, THM. 2.17] this proves that the H(r) moment problem is indeterminate. Thus due to [3], ϕ in (10) can be chosen as absolutely continuous w.r.t. the Borel measure. Then ϕ cannot be extremal solution of the H(r) moment problem (see [8, THM. 2.13]) and the M. Riesz theorem yields $\overline{\Pi_1^{\infty}} \neq L_2(R, \phi)$. Since the conditions $\overline{\Pi_1^{\infty}} = L_2(R,\phi)$ and $\overline{\Pi_n^{\infty}} = L_2(R^n, \phi^{(n)})$, $n>1$, are equivalent (see [6]) we conclude that there exists a nonzero function $\varkappa$ from $L_2(R^n, \phi^{(n)})$ such that

$$\int_{R^n} p \varkappa \, d\phi^{(n)} = 0 \text{ for all } p \in \Pi_n^{\infty}.$$

Therefore

$$\int_{R^n} m_i(\varrho + z\varkappa) \, d\phi^{(n)} = \mu_i, \; i=1,2,\ldots \, ,$$

for any complex number z. This proves (i).

Finally, note that if ϕ is chosen as extremal solution of the H(r) moment problem then ϕ is an atom measure. Moreover $\overline{\Pi_n^{\infty}} = L_2(R^n, \phi^{(n)})$ which means that (10) holds for a unique ϱ. If ϕ is atom and nonextremal then $\overline{\Pi_n^{\infty}} = L_2(R^n, \phi^{(n)})$ and consequently (10) is valid for infinitely many functions ϱ. This proves (ii) and completes the proof. ■

Acknowledgments. I am indebted to Professor H. Woźniakowski for help during the preparation of this paper. My thanks go also to Professor R.P. Boas who pointed out to me that Theorem 1 for n=1

is essentially due to Borel.

References

1. C.Berg, J.P.R.Christensen and C.U.Jensen. A remark on the multidimensional moment problem. Math. Ann. 243, pp. 163-169, 1979.
2. T.S.Chihara. An Introduction to Orthogonal Polynomials. Mathematics and Its Applications, vol. 13, Gordon and Breach, 1978.
3. H.Hamburger. Über eine Erweiterung des Stieltjes'schen Momentproblems. Math. Ann. 81, pp.235-319, 1920.
4. M.A.Kowalski. The recursion formulas for orthogonal polynomials in n-variables. SIAM J. Math. Anal. 13, pp. 309-315, 1982.
5. M.A.Kowalski. Orthogonality and recursion formulas for polnomials in n-variables, SIAM J. Math. Anal. 13, pp. 316-321, 1982.
6. M.A.Kowalski. Ortogonalność a formuły rekurencyjne dla wielomianów wielu zmiennych (Doctoral Thesis). University of Warsaw, 1980.
7. G.Polya. Sur l'indetermination d'un probleme voisin du probleme des moments. C. R. Acad. Sci. Paris 207, pp. 708-711, 1938.
8. A.J.Shohat and J.D.Tamarkin. The Problem of Moments. American Mathematical Society, New York, 1943.

Institute of Informatics, University of Warsaw
P.K.iN. 8 s.850, P.O.Box 1210, 00-901 Warsaw
Poland