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ON THE CONVERGENCE OF TWO-DIMENSIONAL CONTINUED FRACTIONS

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1. Introduction. Correspondence of sequences of meromorphic functions to a formal power series plays a key role in the theory of continued fractions and Padé approximants [1].

It is posible to constract a two-dimensional continued fraction for the formal double power series [2,3] . The arbitrary n-th approximant of this fraction coincides with the first n terms of the initial formal power series in the sense of Padé approximation. Such fractions are the analog of the corresponding ordinary continued fractions[1].

We will establish some conditions for absolute and uniform convergence of the two-dimensional continued fractions.

2. Definitions and properties of the two-dimensional continued fraction. By a two-dimensional continued fraction we mean an expression of the form

$$\cfrac{a_0}{\Phi_0+\cfrac{a_1}{\Phi_1+\dots}} = \frac{a_0|}{|\Phi_0} + \frac{a_1|}{|\Phi_1} + \dots = \sum_{i=0}^{\infty} \frac{a_i|}{|\Phi_i} \qquad ,(1)$$

where

$$\Phi_i = b_i + \sum_{k=1}^{\infty} \frac{a_{k+i,i}|}{|b_{k+i,i}} + \sum_{k=1}^{\infty} \frac{a_{i,k+i}|}{|b_{i,k+i}} .$$

The n-th approximant of the two-dimensional continued fraction (shortly TDCF) (1) is defined as

$$\frac{A_n}{B_n}=\sum_{i=0}^{[\frac{n-1}{2}]}\frac{a_i \mid}{\mid \Phi_i^{(n-1-2i)}}, \quad \Phi_i^{(m)}=b_i+\sum_{k=1}^{m}\frac{a_{k+i,i}\mid}{\mid b_{k+i,i}}+\sum_{k=1}^{m}\frac{a_{i,k+i}\mid}{\mid b_{i,k+i}} . \qquad (2)$$

Here $[x]$ denotes the integral part of x, $\Phi_i^{(0)}=b_i$, $n=1,2,\ldots$.

The other definition of the n-th approximant of TDCF (1) using the composition of linear fractional transformations is given in [4].

TDCF is said to converge or to be convergent if the limit of its sequence of approximants $\lim_{n\to\infty}\{A_n/B_n\}$ exists and is finite. The value of TDCF is defined to be the limit of its sequence of approximants.

TDCF is said to converge absolutely if

$$\sum_{n=0}^{\infty}\left|\frac{A_{n+1}}{B_{n+1}}-\frac{A_n}{B_n}\right|<\infty .$$

Noting

$$Q_i^{(s-1-2i)}=\Phi_i^{(s-1-2i)}+\sum_{j=i+1}^{[\frac{s-1}{2}]}\frac{a_j \mid}{\mid \Phi_j^{(s-1-2j)}}, \quad Q_i^{(0)}=\Phi_i^{(0)}, \quad Q_i^{(-1)}=\infty,$$

$$i=0,1,\ldots,[\tfrac{s-1}{2}], \quad s=1,2,\ldots$$

and using mathematical induction, it is not difficult to obtain the difference formula of the r-th and m-th (r< m) approximants:

$$\frac{A_r}{B_r}-\frac{A_m}{B_m}=\sum_{i=0}^{[\frac{r-1}{2}]}\frac{(-1)^{i+1}\left[\Phi_i^{(r-1-2i)}-\Phi_i^{(m-1-2i)}\right]\prod_{j=0}^{i}a_j}{\prod_{j=0}^{i}Q_j^{(r-1-2j)}\,Q_j^{(m-1-2j)}}+$$

$$+\frac{(-1)^{[\frac{r-1}{2}]}\prod_{i=0}^{[\frac{r-1}{2}]+1}a_i}{\prod_{j=0}^{[\frac{r-1}{2}]}Q_j^{(r-1-2j)}\prod_{j=0}^{[\frac{r-1}{2}]+1}Q_j^{(m-1-2j)}} . \qquad (3)$$

TDCF (1) with complex elements is called a majorant of TDCF

$$\sum_{i=0}^{\infty}\frac{a_i(z)\mid}{\mid \Phi_i(z)} , \qquad (4)$$

where $a_i(z), b_i(z), a_{ij}(z), b_{ij}(z)$, $i \neq j$ - arbitrary complex-valued functions defined in the domain $D \subset C^2$, $z = (z_1, z_2)$ if there exist an integer s and positive constant M, such that for arbitrary integers m,n $\geqslant$ s

$$\left| \frac{P_n}{Q_n} - \frac{P_m}{Q_m} \right| \leq M \left| \frac{A_n}{B_n} - \frac{A_m}{B_m} \right|,$$

P_k / Q_k - the k-th approximant of TDCF (4).

TDCF (4) uniform convergence if the majorant of TDCF (4) converges.

If the elements of TDCF (1) are functions of one or more variables over a certain domain D, then TDCF is said to converge uniformly over D if it converges for all values of the variables in D, and if its sequence of approximants converges uniformly over D.

3. The absolute convergence of TDCF.

Theorem 1. TDCF (1) with complex elements converges absolutely if

$$|b_i| \geqslant |a_i| + 3$$
$$|b_{ij}| \geqslant |a_{ij}| + 1, \quad i \neq j, \quad i,j = 0,1,\ldots \qquad (5)$$

and the set of its values belongs to disk

$$|z| \leq 1 \qquad .(6)$$

Theorem 1 is the analog of Pringsheim's theorem for ordinary continued fractions and its proof contains in [5].

Theorem 2. TDCF (1) with complex elements convergence if

$$\left| \frac{a_0}{b_0} \right| \leq \frac{p_0 - 3}{p_0}, \quad \left| \frac{a_i}{b_{i-1}\, b_i} \right| \leq \frac{p_i - 3}{p_{i-1}\, p_i}, \quad i = 1, 2, \ldots$$

$$\left| \frac{a_{i+1,1}}{b_i\, b_{i+1,1}} \right| \leq \frac{p_{i+1,i} - 1}{p_i\, p_{i+1,i}}, \quad \left| \frac{a_{i,i+1}}{b_i\, b_{i,i+1}} \right| \leq \frac{p_{i,i+1} - 1}{p_i\, p_{i,i+1}}, \qquad (7)$$

$$\left| \frac{a_{i+k,i}}{b_{i+k-1,i}\, b_{i+k,i}} \right| \leq \frac{p_{i+k,i} - 1}{p_{i+k-1,i}\, p_{i+k,i}}, \quad \left| \frac{a_{i,i+k}}{b_{i,i+k-1}\, b_{i,i+k}} \right| \leq \frac{p_{i,i+k} - 1}{p_{i,i+k-1}\, p_{i,i+k}}$$

all p_i, p_{ij}, $i \neq j$ are real and $p_i > 3$, $p_{ij} > 1$.

Proof. Statement of Theorem is obtained by an application of Theorem 1 to the equivalent TDCF

$$\frac{\rho_0 a_0 |}{| \rho_0 \hat{\Phi}_0} + \frac{\rho_0 \rho_1 a_1 |}{| \rho_1 \hat{\Phi}_1} + \frac{\rho_1 \rho_2 a_2 |}{| \rho_2 \hat{\Phi}_2} + \dots ,$$

$$\hat{\Phi}_i = b_i + \left\{ \frac{\rho_{i+1,i}\, a_{i+1,i} |}{| \rho_{i+1,i}\, b_{i+1,i}} + \sum_{k=2}^{\infty} \frac{\rho_{k+i-1,i}\, \rho_{k+i,i}\, a_{k+i,i} |}{| \rho_{k+i,i}\, b_{k+i,i}} \right\} +$$

$$+ \left\{ \frac{\rho_{i,i+1}\, a_{i,i+1} |}{| \rho_{i,i+1}\, b_{i,i+1}} + \sum_{k=2}^{\infty} \frac{\rho_{i,k+i-1}\, \rho_{i,k+i}\, a_{i,k+i} |}{| \rho_{i,k+i}\, b_{i,k+i}} \right\}$$

and then setting $\rho_i = p_i / b_i$, $\rho_{ij} = p_{ij} / b_{ij}$, $i \neq j$, $i, j = 0, 1, \dots$.

Corollary 1. TDCF (1) with partial denominators equal unity converges if

$$|a_0| \le \frac{1}{4}, |a_i| \le \frac{1}{16}, |a_{i+1,i}| \le \frac{1}{8}, |a_{i,i+1}| \le \frac{1}{8}, |a_{i+k,i}| \le \frac{1}{4}, |a_{i,i+k}| \le \frac{1}{4} . \quad (8)$$
$$k = 2, 3, \dots$$

Corollary 2. TDCF (1) with partial numerators equal unity converges if

$$\frac{1}{|b_{2n-1}|} + \frac{3}{|b_{2n}|} \le 1, \; n = 1, 2, \dots$$

$$\frac{1}{|b_{2n-1,k}|} + \frac{1}{|b_{2n,k}|} \le 1 \;, \; \frac{1}{|b_{k,2n-1}|} + \frac{1}{|b_{k,2n}|} \le 1, \; k < 2n-1$$

$$(|b_{2n-1}| + |b_{2n}|)(|b_{2n-1}||b_{2n,2n-1}|)^{-1} \le 1, (|b_{2n-1}| + |b_{2n}|)(|b_{2n-1}||b_{2n-1,2n}|)^{-1} \le 1 \quad . (9)$$

Corollary 1 follows from Theorem 2 if one sets $p_i = 4, p_{ij} = 2, i \neq j$; corollary 2 can be established by setting in (7) $p_{2n-1} = p_{2n} = |b_{2n}|$,

$p_{2n-1,2n} = |b_{2n-1,2n}|$, $p_{2n,2n-1} = |b_{2n,2n-1}|$, $p_{k,2n-1} = p_{k,2n} = |b_{k,2n}|$,

$p_{2n-1,k} = p_{2n,k} = |b_{2n,k}|$, $k < 2n-1$.

4. The uniform convergence of TDCF. In the investigation of the convergence of the even and odd parts of TDCF (1) [5] two-dimensional analogs of well-known g-fractions are appeared [6].

Theorem 3. Let x_i, x_{ij} $(i \neq j, i, j = 0, 1, \dots)$ - complex variables, g_i, g_{ij} $(i \neq j, i, j = 0, 1, \dots)$ such real constant that

$$0 \le g_i < 1 \quad , 0 \le g_{ij} < 1 \qquad (10a)$$

or

$$0 < g_i \le 1 \ , \ 0 < g_{ij} \le 1 . \qquad (10b)$$

Then

1) TDCF

$$\frac{g_0}{\Phi_0 + \sum\limits_{i=1}^{\infty} \frac{(1-g_{i-1})\, g_i\, x_i |}{| \Phi_i}} \qquad (11)$$

$$\Phi_i = 1 + \frac{(1-g_i)\, g_{i+1,i}\, x_{i+1,i}}{1 + \sum\limits_{k=1}^{\infty} \frac{(1-g_{i+k,i})\, g_{i+k+1,i}\, x_{i+k+1,i} |}{| \ 1}} + \frac{(1-g_i)\, g_{i,i+1}\, x_{i,i+1}}{1 + \sum\limits_{k=1}^{\infty} \frac{(1-g_{i,i+k})\, g_{i,i+k+1}\, x_{i,i+k+1} |}{| \ 1}}$$

uniform converges for $|x_{i+1,i}| \le \alpha$, $|x_{i,i+1}| \le \alpha$, $|x_i| \le \beta$, $|x_{ij}| \le 1$, $|i-j| \ne 1$, $i \ne j$, $0 < \alpha < 1/2$, $0 < \beta < 1-2\alpha$.

2) values of TDCF (11) and its approximants belong to the disk

$$|z| \le 1 - 1/s \qquad (12)$$

if the (10a) holds , where

$$s = 1 + \sum_{p=1}^{\infty} \frac{q_0 q_1 \ldots q_p}{(1-q_0)(1-q_1)\ldots(1-q_p)} \ , \quad q_k = \frac{g_k}{1-2\alpha(1-g_k)}$$

moreover the value of the infinitive fraction (11) equals to $1-1/s$, if $x_{i+1,i} = -\alpha$, $x_{i,i+1} = -\alpha$, $x_i = -\beta$, $x_{i+k,i} = x_{i,i+k} = -1$, $k = 2,3,\ldots$.

3) under the assumption (10b) the values of TDCF (11) and its approximants belong to the domain

$$|z - (2-q_0)^{-1}| \le (1-q_0)(2-q_0)^{-1}. \qquad (13)$$

Proof. Using the difference formula (3) it is easy establish that the fraction

$$\frac{q_0 |}{| 1} - \frac{(1-q_0) q_1 |}{| 1} - \frac{(1-q_1) q_2 |}{| 1} - \ldots \qquad (14)$$

is the majorant of TDCF (11). Correctness of statements 2) and 3) follows from correctness of analogous statements for the fraction (14) [6].

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