КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 507—512

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.507-512

ON AN EQUIVALENT NORM IN L_1 WHICH IS UNIFORMLY CONVEX IN EVERY DIRECTION

Denka N. Kutzarova

1. Introduction. A norm on a Banach space X is said to be uniformly convex in every direction (UCED) if the conditions $x_n, y_n, z \in X$, $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + y_n\| \to 2$ and $x_n - y_n = \lambda_n z$ imply $\|x_n - y_n\| \to 0$.

A norm on a Banach space X is locally uniformly rotund (LUR) if the conditions $x, x_n \in X$, $\|x\| = 1$, $\|x_n\| \to 1$, $\|x + x_n\| \to 2$ imply $\|x - x_n\| \to 0$.

Using the construction of an equivalent LUR norm in [1], it is obtained in [3] that every $L_1(S, \Sigma, \mu)$ admits an equivalent norm which is UCED.

The following statement is proved in [2]. Let Y be a LUR Banach space with B_Y its unit ball and X be a Banach space, $F \subset X^*$, F be norming on X. Let $T : Y \to X$ be a bounded linear operator with T(Y) norm dense in X and $T(B_Y)$ be $\sigma(X,F)$ compact. Then X admits an equivalent $\sigma(X,F)$ lower semicontinuous norm which is LUR.

A counter - example shows that it is not possible to obtain a similar general statement for norms which are UCED. In spite of this, using the construction in [2], we may give a rather simpler proof of the main result in [3] that L_1 can be UCED - renormed.

2. Let (S, Σ, μ) be a probability measure, $X = L_1(S, \Sigma, \mu)$ and $U = L_2(S, \Sigma, \mu)$ with their usual norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Denote by B_2 the closed unit ball of U. Let T be the identity mapping of U into X. TB_2 is weakly compact and TU is norm dense in X. Following [2], define for $x \in X$

$$\Psi(x) = \|x\|_1^2 + \sum_{p,q \in N} 2^{-(p+q)} \rho^2(x, \tfrac{p}{q} K),$$

where $K = TB_2$ and $\rho(x, Q) = \inf\{\|x - y\|_1 : y \in Q\}$, $Q \subset X$.

The Minkowski functional of the set $\{x \in X : \Psi(x) \leqq 1\}$ is an equivalent norm on X. Denote it by $|||\cdot|||$.

Theorem 2.1. The norm $|||\cdot|||$ on $X = L_1(S, \Sigma, \mu)$ is UCED.

Since K is w-compact, then for every $x \in X$ there exists a nearest point to x in αK, $\alpha > 0$, and by strict convexity of B_2, this point is unique. If $\rho(x, \alpha K) > 0$, it follows from $u \in \alpha B_2$, $\|x - Tu\|_1 = \rho(x, \alpha K)$ that $\|u\|_2 = \alpha$. Clearly, $\rho(x, \alpha K)$ is a Lipschitz function with constant 1 on $\{\alpha \geqq 0\}$ for fixed $x \in X$. It is easy to prove the following assertion.

Lemma 2.2. Let $x, y \in X$, $\rho(x, \alpha K) > 0$, $u \in U$, $\|u\|_2 = \alpha$, $\|x - Tu\|_1 = \rho(x, \alpha K)$ and $y = Tu + \lambda(x - Tu)$, $\lambda \geqq 0$. Then,

$$\|y - Tu\|_1 = \rho(y, \alpha K).$$

Let now $x_n, y_n \in X$, $|||x_n||| \to 1$, $|||y_n||| \to 1$, $|||x_n + y_n||| \to 2$. By convexity argument (see [2]), for every $p, q \in N$

$$(1) \qquad \lim_n \left[\rho(x_n, \tfrac{p}{q} K) - \rho(y_n, \tfrac{p}{q} K) \right] = 0.$$

Similarly,

$$(2) \qquad \lim_n \left[\rho(x_n, \tfrac{p}{q} K) - \rho(\tfrac{1}{2}(x_n + y_n), \tfrac{p}{q} K) \right. = 0.$$

Consider $\rho(x_n, \alpha K)$ as a sequence of functions on $\{\alpha \geqq 0\}$. Since these functions are decreasing and uniformly bounded, then there exists a subsequence $\{n\}$ such that $\rho(x_n, \alpha K) \to d(\alpha)$ for each $\alpha \geqq 0$. Since $\rho(x_n, \alpha K)$ are Lip - 1, then $d(\alpha)$ is also Lip - 1 and in particular, $d(\alpha)$ is continuous. Analogously, choose a new subsequence $\{n\}$ such that $\rho(y_n, \alpha K) \to d_1(\alpha)$ with $d_1(\alpha)$ continuous. Hence, (1) and the continuity of $d(\alpha)$ and $d_1(\alpha)$ imply $d_1(\alpha) = d(\alpha)$ for each $\alpha \geqq 0$. Similarly, by (2), it is no loss of generality to assume that for a subsequence $\{n\}$ we have also $\rho((x_n + y_n)/2, \alpha K) \to d(\alpha)$, $\alpha \geqq 0$, i.e.

$$(3) \qquad \rho(x_n, \alpha K),\ \rho(y_n, \alpha K),\ \rho((x_n + y_n)/2, \alpha K) \to d(\alpha),\ \alpha \geqq 0.$$

In the following assertion a modification of the technique in [2] is used.

Lemma 2.3. Let $x_n, y_n \in X$ satisfy (3). Let $0 \leqq \alpha < r$, $\delta = d(\alpha) > d(r)$ and $u_n, v_n \in \alpha B_2$, $\|x_n - Tu_n\|_1 = \rho(x_n, \alpha K)$, $\|y_n - Tv_n\|_1 = \rho(y_n, \alpha K)$. Then there exists a subsequence $\{n\}$ so that $\|u_n - v_n\|_2 \to 0$.

Proof. Let B be the closed unit ball of $(X, \|\cdot\|_1)$. Denote by $\|\cdot\|$ the equivalent norm on X, induced by $K_1 = \alpha K + \delta B$.

Choose on the half lines $[Tu_n, x_n)$ and $[Tv_n, y_n)$ points

x_n' and y_n' , respectively, so that $\| x_n' - Tu_n \|_1 = \delta$, $\| y_n' - Tv_n \|_1 = \delta$.
Since $\delta > 0$, Lemma 2.2 implies $\| x_n' - Tu_n \|_1 = \varrho(x_n' , \alpha K)$, $\| y_n' - Tv_n \|_1 = \varrho(y_n' , \alpha K)$ for n large enough , i.e.

(4) $$\varrho(x_n' , \alpha K) = \delta , \qquad \varrho(y_n' , \alpha K) = \delta .$$

By (3) and the form of x_n' and y_n' we have that $\| x_n' - x_n \|_1 \to 0$, $\| y_n' - y_n \|_1 \to 0$. By this and (3) we obtain $\varrho((x_n' + y_n')/2, \alpha K) \to \delta$.
This and (4) give

(5) $$\| x_n' \| = 1 , \quad \| y_n' \| = 1 , \quad \|(x_n' + y_n')/2 \| \to 1 .$$

Let $f_n \in X^*$ be such that $\| f_n \| = 1$ and

(6) $$f_n(x_n' + y_n') = \| x_n' + y_n' \| .$$

Therefore, (5) and (6) imply

(7) $$f_n(x_n') \to 1 , \quad f_n(y_n') \to 1 .$$

Put $w_n = x_n' - Tu_n$. By $\| f_n \| = 1$ and (7) , we have

$$\lim_n (\sup_{K_1} f_n - f_n(Tu_n + w_n))$$

$$= \lim_n (\sup_{\alpha K} f_n - f_n(Tu_n)) + \lim_n (\sup_{\delta B} f_n - f_n(w_n)) = 0 .$$

Since both summands are non-negative, it follows that

(8) $$\lim_n (\sup_{\alpha K} f_n - f_n (Tu_n)) = 0 .$$

Similarly,

(9) $$\lim_n (\sup_{\alpha K} f_n - f_n (Tv_n)) = 0 .$$

We shall show that $\liminf_n (\sup_{\alpha K} f_n)$ 0. It follows from $B_\delta \subset K_1$ and $\| f_n \| = 1$ that

(10) $$\sup (f_n(x) : \| x \|_1 \leqq 1) \leqq 1/\delta .$$

We conclude from $\| x_n' - x_n \|_1 \to 0$ and $\| y_n' - y_n \|_1 \to 0$ that x_n' , y_n' , $(x_n' + y_n')/2$ satisfy (3). Hence, by $\varrho(x_n', rK) \to d(r)$, there exist $z_n \in rK$ so that $\| x_n' - z_n \|_1 < \delta - 2\delta_1$ for n sufficiently large , where $\delta_1 = (\delta - d(r))/3 > 0$. This, (7) and (10) give for n sufficiently large

$$\sup_{rK} f_n \geqq | f_n(z_n)| \geqq | f_n(x_n')| - | f_n(z_n - x_n')|$$

$$\geqq | f_n(x_n')| - (\delta - 2\delta_1)/\delta > \delta_1/\delta > 0 .$$

Therefore, there exists a subsequence $\{n\}$ such that

(11) $$\lim_n (\sup_{\alpha K} f_n) = t > 0 .$$

Define h_n , h_n' U^*, $h_n'(u) = f_n(Tu)$, $h_n = h_n' / \|h_n'\|_2$.

It follows from (8) , (9) and (11) that $f_n(Tu_n) \to t$, $f_n(Tv_n) \to t$.

Moreover , by (11) , $\|h_n'\|_2 \to t / \alpha$. Thus, $h_n(u_n) \to \alpha$, $h_n(v_n) \to \alpha$.

Consequently,

$$\liminf_n \| u_n + v_n \|_2 = \liminf_n h_n(u_n + v_n) = 2\alpha .$$

By this, $\|u_n\|_2 = \|v_n\|_2 = \alpha$ and the uniform convexity of U , we obtain that

$$\| u_n - v_n \|_2 \to 0 .$$

2.4. Proof of Theorem 2.1. It suffices to show that the conditions x_n , y_n , $z \in X$, $|||x_n||| \to 1$, $|||y_n||| \to 1$, $|||x_n + y_n||| \to 2$, $x_n - y_n = z$, $n = 1,2,\ldots$ imply $z = 0$.

Choose a subsequence $\{n\}$ to satisfy (3). Assume that $z \neq 0$. Put $\|z\|_1 = \varepsilon$, $\varepsilon > 0$. There exists an $\eta > 0$ so that $\mu (C) < \eta$ implies $\| z \chi_C \|_1 < \varepsilon / 2$. Denote

$$C_n = \{ | x_n | > 3/\eta \} \cup \{ | y_n | > 3/\eta \} .$$

Thus, $\limsup_n \mu (C_n) < \eta$, whence $\mu (C_n) < \eta$ for n sufficiently large. Put $A_n = S \setminus C_n$. Therefore, $\| z \chi_{A_n} \|_1 > \varepsilon/2$, i.e.

(12) $$\|(x_n - y_n) \chi_{A_n} \|_1 > \varepsilon / 2 \quad \text{for } n \text{ sufficiently large}$$

By this, there exists a subsequence $\{n\}$ such that either for $\{x_n\}$ or for $\{y_n\}$ - say for $\{x_n\}$, the inequalities $\| x_n \chi_{A_n} \|_1 \geq \varepsilon/4$ hold. It follows from $| x_n \chi_{A_n} | \leq 3/\eta$ that $\| x_n \chi_{A_n} \|_2 \leq 3/\eta$.

We have

$$\| x_n \|_1 = \| x_n - x_n \chi_{A_n} \|_1 + \| x_n \chi_{A_n} \|_1 \geq \rho (x_n , 3 \eta^{-1} K) + \varepsilon / 4 .$$

Therefore, $d(0) \geq d(3/\eta) + \varepsilon/4$, whence

(13) $$d(0) > \lim_{\alpha \to +\infty} d(\alpha) = d .$$

Since $d(\alpha)$ is continuous and decreasing , according to (13) , we may select α and r such that $0 < \alpha < r$ and

$$d + \varepsilon/6 > d(\alpha) > d(r) .$$

Let u_n, $v_n \in \alpha B_2$, $\| x_n - Tu_n \|_1 = \rho (x_n , \alpha K)$ and $\| y_n - Tv_n \|_1 = \rho (y_n , \alpha K)$.

Then, it follows from Lemma 2.3 that there exists a subsequence $\{n\}$ so that $\|u_n - v_n\|_2 \to 0$, whence

$$(14) \qquad \|Tu_n - Tv_n\|_1 \to 0 .$$

We have that

$$\|x_n - Tu_n\|_1 = \|(x_n - Tu_n)\chi_{C_n}\|_1 + \|(x_n - Tu_n)\chi_{A_n}\|_1 .$$

Moreover,

$$(x_n - Tu_n)\chi_{C_n} = x_n - (x_n\chi_{A_n} + Tu_n\chi_{C_n}) = x_n - T(x_n\chi_{A_n} + u_n\chi_{C_n})$$

and $\|x_n\chi_{A_n} + u_n\chi_{C_n}\|_2 \leq 3/\eta + \alpha = \eta_1$. Then,

$$d \leq d(\eta_1) \leftarrow \rho(x_n, \eta_1 K) \leq \|(x_n - Tu_n)\chi_{C_n}\|_1$$

$$= \|x_n - Tu_n\|_1 - \|(x_n - Tu_n)\chi_{A_n}\|_1 = \rho(x_n, \alpha K) - \|(x_n - Tu_n)\chi_{A_n}\|_1 .$$

Therefore,

$$d \leq d(\alpha) - \liminf_n \|(x_n - Tu_n)\chi_{A_n}\|_1 < d + \varepsilon/6 - \liminf_n \|(x_n - Tu_n)\chi_{A_n}\|_1 ,$$

i.e.

$$\liminf_n \|(x_n - Tu_n)\ {}_{A_n}\|_1 < \varepsilon/6 .$$

Select a subsequence $\{n\}$ so that

$$\|(x_n - Tu_n)\chi_{A_n}\|_1 < \varepsilon/6 .$$

Similarly, we may choose a new subsequence $\{n\}$ such that

$$\|(y_n - Tv_n)\chi_{A_n}\|_1 < \varepsilon/6 .$$

It follows from (14) that $\|Tu_n - Tv_n\|_1 < \varepsilon/6$ for n sufficiently large. Then,

$$\|(x_n - y_n)\chi_{A_n}\|_1 < \varepsilon/2 ,$$

which contradicts (12). Hence, $z = 0$, which completes the proof.

References

1. W.Davis, N.Ghoussoub and J.Lindenstrauss. A lattice renorming theorem and applications to vector-valued processes. Trans. Amer. Math. Soc., 263, 1981, 531 - 540.
2. G.Godefroy, S.Troyanski, J.H.M.Whitfield, V.Zizler. Smoothness in Weakly Compactly Generated Banach spaces. J.Funct. Analysis, 52, 1983, 344 - 352.
3. D.N.Kutzarova and S.L.Troyanski. On equivalent lattice norms which are uniformly convex or uniformly differentiable in every direction in Banach lattices with a weak unit. Serdica, 9, 1983, 249 - 262.

Institute of Mathematics with Computer Center
Bulgarian Academy of Sciences
P.O.Box 373 1090 Sofia Bulgaria