КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 513—518

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.513-518

A NOTE ON GENERALIZED QUADRATURE FORMULAS OF GAUSS-JACOBI TYPE

Guillermo López Lagomasino
Jesús Illán

1. Introduction. Let μ be a positive Borel measure whose support S consists of an infinite set of points contained in the finite real segment $[a,b]$ and Ω a region in the complex plane $\mathbb{C}$ symmetric with respect to the real line $\mathbb{R}$ such that $[a,b]\subset\Omega$. Let $\alpha = \{\alpha_{n,1},\dots,\alpha_{n,2n}\}$, $n\in\mathbb{N}$, be a triangular table of points, $\alpha\subset\overline{\mathbb{C}}\setminus\Omega$, such that each row $(\alpha_{n,1},\dots,\alpha_{n,2n})$ is also symmetric with respect to $\mathbb{R}$ and $w_{2n}(z) = \prod_{k=1}^{2n}(z-\alpha_{n,k})$. If $\alpha_{n,k}=\infty$, for some k, then we consider the corresponding factor in w_{2n} as equal to 1. Since w_{2n} has no zeros in $[a,b]$ and all its coefficients are real then w_{2n} doesn't change sign on $[a,b]$; we will assume in the following, without loss of generality, that $w_{2n}(x)>0$ on $[a,b]$. By $Q_n(z) = \prod_{k=1}^{n}(z-x_{n,k})$ we will denote the n-th orthogonal polynomial with respect to the positive Borel measure determined by $\frac{d\mu}{w_{2n}} = d\mu_n$. Obviously, all the zeros of Q_n are simple and lie on $[a,b]$.

If $\alpha_{n,k}\equiv\infty$, then $w_{2n}(z)\equiv 1$, $d\mu_n = d\mu$ for all $n\in\mathbb{N}$, and $\{Q_n\}$, $n\in\mathbb{N}$, is the sequence of monic orthogonal polynomials with respect to μ. Let f be an arbitrary continuous function on S. In this case, the approximation of $\int f\,d\mu$ by means of quadrature formulas of Gauss-Jacobi $\sum_{k=1}^{n}\lambda_{n,k}f(x_{n,k})$, where $\lambda_{n,k}$ is the Christoffel coefficient of Q_n at $x_{n,k}$, has been well studied. In [1], we have studied this problem for general w_{2n} and obtained convergence results which extend classical ones of Stieltjes. In this paper, we will restrict our attention to classes of holomorphic functions on [a,b] and we will estimate the rate of convergence of the respective generalized quadrature formulas of Gauss-Jacobi type.

In the following, we will maintain the notations introduced above.

2. Auxiliary results. The proof of the following lemma is analogous to the classical case (see [2, p. 47]).

Lemma 1 Let $\mathcal{P}$ be an arbitrary polynomial of degree $\leq 2n-1$, then

$$\int \left(\frac{\mathcal{P}}{w_{2n}}\right)(x)\,d\mu(x) = \sum_{k=1}^{n} \lambda_{n,k} \left(\frac{\mathcal{P}}{w_{2n}}\right)(x_{n,k}), \tag{1}$$

where

$$0 < \lambda_{n,k} = \int \left[\frac{Q_n(x)}{Q_n'(x_{n,k})\,(x-x_{n,k})}\right]^2 \frac{w_{2n}(x_{n,k})}{w_{2n}(x)}\,d\mu(x). \tag{2}$$

Making use of the previous lemma it is easy to prove the following (see [3])

Lemma 2 Let $\hat{\mu} = \mu * \frac{1}{z}$, then for all $n \in \mathbb{N}$ there exists a unique rational function $R_n = \frac{p_{n-1}}{q_n}$, $\deg p_{n-1} \leq n-1$, $\deg q_n \leq n$, such that

$$(\hat{\mu} - R_n)(z) = \frac{w_{2n}(z)}{Q_n^2(z)} \int \frac{Q_n^2(x)}{w_{2n}(x)} \frac{d\mu(x)}{z-x} \tag{3}$$

Moreover,

$$R_n(z) = \sum_{k=1}^{n} \frac{\lambda_{n,k}}{z-x_{n,k}} \quad (q_n \equiv Q_n). \tag{4}$$

Formula (3) guarantees that $\frac{\hat{\mu} - R_n}{w_{2n}}$ is analytic in $\overline{\mathbb{C}} \setminus [a,b]$ and in a neighborhood of $z = \infty$

$$(\hat{\mu} - R_n)(z) = \frac{A_n}{z^{2n+1}} + \dots , \tag{5}$$

where on the right hand side of (5) stands a series on increasing powers of $\frac{1}{z}$. We conclude that R_n is the multipoint Padé aproximant of type (n,n) which interpolates to $\hat{\mu}$ at the points $\alpha_{n,k}$, k=1, 2, ..., 2n. (for more details see [3]).

In the following, for simplicity, we will consider that Ω is a bounded region; that is, ∞ is an interior point of $\overline{\mathbb{C}} \setminus S$.

Lemma 3 Let f be an analytic function on Ω then

$$\int f(x)d\mu(x) - \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k}) =$$

$$\frac{1}{2\pi i} \int_{\Gamma} f(z)[(\hat{\mu} - R_n)(z)]dz, \qquad (6)$$

where Γ is an arbitrary contour in Ω that surrounds $[a,b]$.

The proof is straightforward using Cauchy's integral formula, Fubini's theorem and (4).

Let's denote by $g_S(z,\xi)$ the generalized Green function with respect to the region $\overline{\mathbb{C}} \setminus S$ with singular point at $z = \xi$. For any pair of compact sets $K \subset \overline{\mathbb{C}} \setminus S$ and $E \subset \overline{\mathbb{C}} \setminus S$ let's put

$$\sigma = \sigma(S,K,E) = \inf \{ g_S(z,\xi) : z \in K, \xi \in E \}.$$

If $S \subset F$, where F is the union of a finite number of segments and Ω is a regular region with respect to Dirichlet's problem, let's denote by h(z) the harmonic measure on $\Omega \setminus F$ such that $h|_{\partial\Omega} \equiv 1$ and $h|_F \equiv 0$. The value

$$C = C(\overline{\mathbb{C}} \setminus \Omega, F) = \frac{1}{2\pi} \int_{\Gamma} \frac{\partial h}{\partial \eta} ds$$

(Γ is an arbitrary contour in $\Omega \setminus F$ that separates $\overline{\mathbb{C}} \setminus \Omega$ from F, $\frac{\partial}{\partial \eta}$ is the normal derivative in the direction from F to $\overline{\mathbb{C}} \setminus \Omega$ and ds is the arc element) is called the capacity of the condenser

$(\overline{\mathbb{C}} \setminus \Omega, F)$. The table $\alpha \subset \overline{\mathbb{C}} \setminus \Omega$ is said to be extremal with respect to the condenser $(\overline{\mathbb{C}} \setminus \Omega, F)$ if

$$\frac{1}{2n} \sum_{k=1}^{2n} g_F(z, \alpha_{n,k}) \rightrightarrows \frac{h(z)}{C}, \quad z \in \Omega \setminus F,$$

where the double arrow stands for uniform convergence on each subset of $\Omega \setminus F$. Such tables habe been well studied in the context of rational approximation with fixed poles to analytic functions (see [4], chapters VIII and IX).

In [3], the following estimates were proved:

$$\overline{\lim} \| \hat{\mu} - R_n \|_K^{1/2n} \leq e^{-\sigma}, \quad K \subset \overline{\mathbb{C}} \setminus [a,b], \qquad (7)$$

where $\sigma = \sigma(S, K, \alpha')$ and α' as usual stands for the set of limit points of α ; if $S \subset F$, Ω is regular and α is extremal with respect to $(\overline{\mathbb{C}} \setminus \Omega, F)$ then

$$\overline{\lim} \| \hat{\mu} - R_n \|_K^{1/2n} \leq \exp\left(- \frac{\tau(K)}{C} \right) \qquad (8)$$

where $\tau(K) = \inf \{ h(z) : z \in K \}$.

<u>3. Main results</u>. Using lemma 3 and the estimates (7) and (8) immediately follow:

<u>Theorem 1</u>. Let μ be a positive Borel measure whose support S consists of an infinite set of points contained in $[a,b] \subset \mathbb{R}$, f is an analytic function in the region $\Omega \supset [a,b]$ and $\alpha = \{\alpha_{n,k}\}$, $k = 1, 2, \ldots, 2n$, $n \in \mathbb{N}$, is a triangular table of points contained in $\overline{\mathbb{C}} \setminus \Omega$. Then

$$\overline{\lim} \left| \int f(x) d\mu(x) - \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k}) \right|^{1/2n} \leq e^{-\sigma},$$

where $\{x_{n,k}\}$, $k=1, \ldots, n$, is the set of zeros of the n-th orthogonal polynomial Q_n respect to $\frac{d\mu}{w_{2n}}$, $\lambda_{n,k}$ is (given by (2)) the generalized Christoffel coefficient of Q_n at $x_{n,k}$, and $\sigma = \sigma(S, \partial\Omega, \alpha')$.

Theorem 2. Under the conditions of theorem 1, if $S = F$ is the union of a finite number of segments, Ω is regular and α is extremal with respect to the condenser $(\overline{\mathbb{C}} \setminus \Omega, F)$, then

$$\overline{\lim} \left| \int f(x) d\mu(x) - \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k}) \right|^{1/2n} \leq \exp\left(-\frac{1}{C}\right), \qquad (9)$$

where $C = C(\overline{\mathbb{C}} \setminus \Omega, F)$.

Under the conditions of theorem 2, if $S \subset F$ then we can approximate S by a decreasing sequence of sets $\{F_m\}$, $m \in \mathbb{N}$, $(F_m \supset F_{m+1}, \bigcap_{m \in \mathbb{N}} F_m = S)$ such that for each $m \in \mathbb{N}$, F_m is the union of a finite number of segments. In this case, we can obtain an estimate of type (9) taking $C = C(\overline{\mathbb{C}} \setminus \Omega, S)$. On the other hand, by means of bilinear transformations it's easy to see that the above results do not depend on whether Ω is a bounded or unbounded set.

So, if we want to approximate $\int f d\mu$, where f is analytic on $\Omega \supset [a,b]$, a good choise, as (9) shows, is to take a table extremal with respect to $(\overline{\mathbb{C}} \setminus \Omega, F)$, where F is as "near" as possible to S, and build up the corresponding generalized quadrature formulas of Gauss-Jacobi type.

References

1. G. López and J. Illán, Formulas de cuadratura para intervalos no acotados, Rev. Ciencias Mat., v. III (1982), No. 3.

2. G. Szego, Orthogonal Polynomials, Coll. Pub. A.M.S., v. 23, Providence, 1939.

3. А.А. Гончар, Г. Лопес, О теореме Маркова для многоточечных аппроксимаций Паде, Мат. Сб., 105(147)(1978), 4, 512-524.

4. J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, 5th ed. Coll. Pub. A.M.S., v. XX, Providence, 1969.