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A SPLINE INTERPOLATIONAL METHOD FOR FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. In this work we introduce a spline interpolational method for functions of several variables defined on an n-dimensional rectangle. The interpolating spline function is a piecewise polynomial of degree at most 2 in each variable. The coefficients of the polynomials are expressed by simple explicit formulas containing the function values at the lattice points of a subdivision of the rectangle. Theorem 2 shows that this method gives a uniform approximation for continuous functions together with the uniform approximations of the derivates for C^1-, and C^2-functions.

A simple application of the method is given to approximate multiple integrals by a quadrature formula, which is illustrated on a numerical example. In contrast with the Simpson formula the estimation of the error is very easy, even for functions which are continuous only. [1]

For the sake of simplicity we introduce some notations concerning functions of several variables.

Let $\mathbb{R}^n$ denote the n dimensional euclidean space. For $a, b \in \mathbb{R}^n$ we let $a \le b$ if and only if $a_{(j)} \le b_{(j)}$, $j=1,\dots,n$ ($a_{(j)}$ is the j-th coordinate of a). If $a, b \in \mathbb{R}^n$, then $[a,b] = \{x \in \mathbb{R}^n : a \le x \le b\}$, further $a \circ b \in \mathbb{R}^n$ with $(a \circ b)_{(j)} = a_{(j)} b_{(j)}$. For $h, k \in \mathbb{R}^n$ let $h^k = \prod_{j=1}^{n} h_{(j)}^{k_{(j)}}$, $|h| = \sum_{j=1}^{n} h_{(j)}$ and $\|h\|$ is the euclidean norm of h. Let θ denote the zero vector in $\mathbb{R}^n$, further $e = (1,1,\dots,1)$ and $e_r = (0,\dots,\overset{r}{1},\dots,0)$, that is $e_{(j)} = 1$, $(e_r)_{(j)} = \delta_{rj}$ $(j, r = 1,\dots,n)$. If $u : \mathbb{R}^n \to \mathbb{R}$, $h, k \in \mathbb{R}^n$, then let Δ denote the difference operator

$$\Delta_h^{(k)} u(t) = \Delta_{h_{(1)},\dots,h_{(n)}}^{k_{(1)},\dots,k_{(n)}} u(t_{(1)},\dots,t_{(n)}) \qquad (t \in \mathbb{R}^n)$$

and $\omega_r(\cdot\,; u)$ is the modulus of continuity of the r-th derivates.

2. Construction of the spline function. Let $a, b \in \mathbb{R}^n$ and $u : [a,b] \to \mathbb{R}$ be a function. Let $\{t_i\}_{\theta \le i \le N}$ be an equidistant subdivision of $[a,b]$ with $h = (h_{(1)}, \dots, h_{(n)})$. Hence $h_{(j)} = (b_{(j)} - a_{(j)})/N_{(j)}$ and in what follows we suppose, that $h_{(j)} \le 1$ and $0 < \alpha \le h_{(j)}/h_{(\ell)} \le \beta$ $(j, \ell = 1, \dots, n)$. We introduce the notation $u_i = u(t_i)$ $(\theta \le i \le N)$. Let the spline function S (corresponding to the function u and to this subdivision) be defined for $t_i \le t \le t_{i+e}$ as follows:

$$S(t) = S_i(t) \qquad (\theta \le i \le N-e), \tag{1}$$

where

$$S_i(t) = u_i + \sum_{j=1}^{n} \Big[\frac{1}{2h_{(j)}} \left(u_{i+e_j} - u_{i-e_j}\right)(t-t_i)_{(j)} + \tag{2}$$
$$+ \frac{1}{2h_{(j)}^2} \Delta_h^{(2e_j)} u_{i-e_j} (t-t_i)_{(j)}^2 \Big] + \sum_{\substack{\theta \le k \le e \\ |k| \ge 2}} \frac{1}{h^k} \Delta_h^{(k)} u_i \, (t-t_i)^k$$

and

$$(e \le i \le N-e)$$

$$S_i(t) = S_{i+e\circ\delta}(t) \qquad (\theta \le i \le N-e,\ e \not\le i) \tag{3}$$

where $\delta_{(j)} = \begin{cases} 1, & i_{(j)} < N_{(j)} - 1 \\ 0, & i_{(j)} = N_{(j)} - 1 \end{cases}$.

This spline function S interpolates u at the lattice points t_i, as the next theorem shows:

THEOREM 1. The functions S_i defined by (2)-(3) satisfy for $\theta \le i \le N-e$

$$S_i(t_i + \ell \circ h) = u_{i+\ell} \qquad (\theta \le \ell \le e).$$

This statement can be verified by an easy computation using the identity

$$\sum_{\theta \le k \le N} \binom{N_{(1)}}{k_{(1)}} \cdots \binom{N_{(n)}}{k_{(n)}} \Delta_h^{(k)} u(t) = u(t + N \circ h).$$

THEOREM 2. If $u\ [a,b] \to \mathbb{R}$ is a p-times continuously differentiable function $(p = 0, 1, 2)$, then we have

$$\sup_{t \in [a,b]} \| \partial_j^r \partial_\ell^s u(t) - \partial_j^r \partial_\ell^s S(t) \| \le C_p \|h\|^{p-r-s} \omega_p(\|h\|; u)$$

$(r, s = 0, 1, 2,\ r+s \le p,\ j, \ell = 1, \dots, n)$, where C_p is independent of h.

PROOF. The proof is very similar in the different cases, and hence, for the sake of brevity we present the proof in one case only, namely for $p=2$, $r=s=0$. If $e\leq i\leq N-e$, then for $t\in[t_i, t_{i+e}]$ we have

$$\|u(t)-S(t)\| = \|u(t)-S_i(t)\| \leq \sum_{j=1}^{n} \|\partial_j u_i - \frac{1}{2h_{(j)}}(u_{i+e_j}-u_{i-e_j})\| h_{(j)} +$$

$$+\frac{1}{2}\sum_{j=1}^{n}\|\partial_j^2 u(\tau)-\partial_j^2 u_i\| h_{(j)}^2 + \frac{1}{2}\sum_{j=1}^{n}\|\partial_j^2 u_i - \frac{1}{h_{(j)}^2}\Delta_h^{(2e_j)}u_{i-e_j}\| h_{(j)}^2 +$$

$$+\sum_{\substack{1\leq j,\ell\leq n\\ j\neq\ell}}\|\partial_j\partial_\ell u(\tau)-\partial_j\partial_\ell u_i\| h_{(j)}h_{(\ell)} + \sum_{\substack{1\leq j,\ell\leq n\\ j\neq\ell}}\|\partial_j\partial_\ell u_i - \frac{1}{h_{(j)}h_{(\ell)}}\Delta_h^{(e_j+e_\ell)}u_i\| h_{(j)}h_{(\ell)} +$$

$$+\sum_{\substack{\theta\leq k\leq e\\ |k|>2}}\|\Delta_h^{(k)}u_i\| \leq C\cdot\|h\|^2\omega_2(\|h\|;u) .$$

In the case $\theta\leq i\leq N-e$, $e\nleq i$ we can proceeed a similar computation. Here we used the estimates for $j,\ell=1,\dots,n$ $(j\neq\ell)$

$$\|\frac{1}{2h_{(j)}}(u_{i+e_j}-u_{i-e_j})-\partial_j u_i\| \leq h_{(j)}\,\omega_2(\|h\|;u)$$

$$\|\frac{1}{h_{(j)}^2}\Delta_h^{(2e_j)}u_{i-e_j}-\partial_j^2 u_i\| \leq \omega_2(\|h\|;u)$$

$$\|\frac{1}{h_{(j)}h_{(\ell)}}\Delta_h^{(e_j+e_\ell)}u_i-\partial_j\partial_\ell u_i\| \leq \left(1+\frac{h_{(j)}}{h_{(\ell)}}+\frac{h_{(\ell)}}{h_{(j)}}\right)\omega_2(\|h\|;u)$$

and for $\theta\leq k\leq e$, $|k|>2$

$$\|\Delta_h^{(k)}u_i\| \leq \text{const.}\ \|h\|^2\,\omega_2(\|h\|;u) \qquad (e\leq i\leq N-e)$$

These estimates are based on the Taylor-formula in several variables.

3. Application: Approximation of multiple integrals.

From Theorem 2 it follows:

THEOREM 3. If $u:[a-h;b]\to\mathbb{R}$ is a p-times continuously differentiable function $(p=0,1,2)$, then we have

$$\left|\int_{[a,b]} u(t)\,dt - \int_{[a,b]} S(t)\,dt\right| = \theta(\|h\|^p)$$

This theorem shows, that the integral of the spline function S provides a good approximation for the integral of the function u. As the integral of S can be computed easily by (1)-(3), we get a numerical quadrature formula for multiple integrals. Indeed, we have:

$$\int_{[a,b]} S(t)\,dt = \sum_{0\le i\le N-e} \int_{[t_i,t_{i+e}]} S_i(t)\,dt =$$

$$= \prod_{j=1}^{n} h_{(j)} \sum_{0\le i\le N-e} \left[\frac{3-n}{3} u_i + \frac{1}{12}\sum_{j=1}^{n}\left(5u_{i+e_j} - u_{i-e_j}\right) + \sum_{\substack{0\le k\le e\\ |k|\ge 2}} \frac{1}{2^{|k|}} \Delta_h^{(k)} u_i\right].$$

It is easy to see, that this quadrature formula produces the exact value if u is a polynomial of degree at most two in each variable.

Here we present the formulas obtained in this way in the cases $n=2,3$.

Two dimensional case. ($n=2$)

Let us denote $h_{(1)}=h$, $h_{(2)}=l$, $N_{(1)}=N$, $N_{(2)}=M$, $i_{(1)}=i$, $i_{(2)}=j$. Then the approximate value of the integral is

$$(4)\qquad \int_{[a,b]} S(t)\,dt = \frac{hl}{12}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left(3u_{i+1,j+1} + 2u_{i+1,j} + 2u_{i,j+1} + 7u_{i,j} - u_{i,j-1} - u_{i-1,j}\right).$$

Three dimensional case. ($n=3$)

Let us denote $i_{(1)}=i$, $i_{(2)}=j$, $i_{(3)}=k$, $N_{(1)}=N$, $N_{(2)}=M$, $N_{(3)}=K$. Then the approximate value of the integral is

$$(5)\qquad \int_{[a,b]} S(t)\,dt = \frac{h_{(1)}h_{(2)}h_{(3)}}{24}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\sum_{k=0}^{K-1}\Big(3u_{i+1,j+1,k+1} + 3u_{i+1,j+1,k} + 3u_{i,j+1,k+1} +$$

$$+ 3u_{i+1,j,k+1} + u_{i+1,j,k} + u_{i,j+1,k} + u_{i,j,k+1} + 15u_{i,j,k} - 2u_{i,j,k-1} - 2u_{i,j-1,k} -$$

$$- 2u_{i-1,j,k}\Big).$$

Example. Here we apply the two dimensional formula for the approximation of the integral

$$\int_0^1\int_{-1}^0 x e^{xy}\,dx\,dy\ .$$

The results are summarized in the following table.

Table. The exact value is: $e^{-1} = 0.3678794$

N	M	Approximate value	Error
10	10	0.36798159	0.0001021
10	15	0.36794945	0.0000699
15	10	0.36792876	0.0000492
20	20	0.36789207	0.0000125
20	25	0.36788952	0.0000100
25	20	0.36788782	0.0000083
30	30	0.36788317	0.0000036
40	10	0.36789422	0.0000147
50	10	0.36788247	0.0000029
60	20	0.36788191	0.0000024

The calculations were proceeded by a Sinclair ZX 81 personal computer.

References

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