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ON THE ALMOST STRONG SUMMABILITY AND CONVERGENCE OF FOURIER SERIES

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Let $L^p_{2\pi}$ $(1 \leqq p < \infty)$ [resp. $C_{2\pi}$] be the class of all real-valued functions, 2π-periodic, Lebesgue-integrable with p-th power [continuous], and let Lipα $(0 < \alpha \leqq 1)$ denote the class of all functions f belonging to $C_{2\pi}$ such that their moduli of continuity satisfy the condition $\omega(\delta) = \omega(\delta; f) = O(\delta^{\alpha})$ as $\delta \to 0+$.

Consider the Fourier series

$$S[f](x) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}\left(a_k(f)\cos kx + b_k(f)\sin kx\right),$$

and denote by $S_n[f]$ the partial sums of $S[f]$, and by

$$\sigma_{m,n}[f] = \frac{1}{m+1}\sum_{k=n}^{n+m} S_k[f]$$

the generalized de la Vallée-Poussin means of the sequence $\{S_n[f]\}$. The sequence $\{S_n[f]\}$ (or series $S[f]$) is said to be almost convergent to f, at a point x, if

$$\sigma_{m,n}[f](x) - f(x) = o(1) \quad \text{as} \quad m \to \infty$$

uniformly with respect to $n=0,1,2,\ldots$ (G.G.Lorentz [2]).

The aim of this note is to investigate the following strong means

$$T_{m,n}[f]_q(x) = \left\{ \frac{1}{m+1} \sum_{\mu=0}^{m} \left| \sigma_{\mu,n}[f](x) - f(x) \right|^q \right\}^{1/q},$$

$$H_{m,n}[f]_q(x) = \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} \left| \sigma_{m,\nu}[f](x) - f(x) \right|^q \right\}^{1/q}$$

for $q > 0$ and $m,n = 0,1,2,\ldots$

Basing on the above Lorentz definition we introduce the following one : the series $S[f]$ is said to be almost $(C,1)$-strong summable of the order q to f, at a point x, i.e. $[C,1]^a_q$ -summable, if

$$T_{m,n}[f]_q(x) = o(1) \quad \text{as} \quad m \longrightarrow \infty$$

uniformly with respect to n ; the sequence $\{S_n[f]\}$ (or the series $S[f]$) is said to be almost convergent in the $[C,1]_q$-mean to f, at a point x, i.e. $[C,1]^a_q$ - convergent, if

$$H_{m,n}[f]_q(x) = o\ (1) \quad \text{as} \quad m \longrightarrow \infty$$

uniformly with respect to n.

The above two relations for $f \in L^p_{2\pi}$ $(p > 1)$ can be treated as some immediate consequences of the Carleson-Hunt theorem [5] , and hold for almost every x .

For the first quantity, by Leindler's theorems [1] , the more precise results may be deduced. Namely

Theorem 1. If $f \in L^1_{2\pi}$, then, for all $x \in (-\infty, \infty)$ and $q > 0$,

$$T_{m,n}[f]_q(x) \leqq C_1 \left\{ \frac{1}{m+1} \sum_{\mu=0}^{m} \left(\frac{1}{\mu+1} \sum_{k=0}^{\mu} w_x\left(\frac{\pi}{k+1}\right) + \sum_{k=\mu+1}^{\mu+1+n} \frac{1}{k+1} w_x\left(\frac{\pi}{k+1}\right) \right)^q \right\}^{1/q}$$

$(n,m=0,1,2,\dots)$, where C_1 is an absolute constant and

$$w_x(\delta)_p = w_x(\delta;f)_p = \sup_{0<h\leq\delta}\left\{\frac{1}{h}\int_0^h |f(x+t)-f(x)|^p\,dt\right\}^{1/p}.$$

From this theorem we obtain the following two corollaries.

Corollary 1. If $f\in \mathrm{Lip}\,\alpha$, then, for $q>0$, the relations

$$\max_x T_{m,n}[f]_q(x) = \begin{cases} O(m^{-\alpha}) & \text{when } \alpha q<1,\ 0<\alpha<1,\\ O(m^{-\alpha}\log^{\alpha} m) & \text{when } \alpha q=1,\ 0<\alpha<1,\\ O(m^{-1/q}) & \text{when } \alpha q>1,\ 0<\alpha\leq 1,\\ O(m^{-1}\log m) & \text{when } q<1,\ \alpha=1,\\ O(m^{-1}\log^2 m) & \text{when } q=1,\ \alpha=1,\end{cases}$$

hold uniformly with respect to n.

Corollary 2. If $f\in L^1_{2\pi}$ and $n=O(1)$, then, for $x\in(-\infty,\infty)$ and $q>0$,

$$T_{m,n}[f]_q(x)\leq C_2\left\{\frac{1}{m+1}\sum_{\mu=0}^{m}\left(\frac{1}{\mu+1}\sum_{k=0}^{\mu} w_x\left(\frac{\pi}{k+1}\right)_1\right)^q\right\}^{1/q}\quad (m=0,1,2,\dots).$$

Next, the estimations of the quantity $H_{m,n}[f]_q$ will be given.

Theorem 2. If $f\in L^1_{2\pi}$, then, for $x\in(-\infty,\infty)$ and $q\in(0,2\rangle$, the inequality

$$H_{m,n}[f]_q(x)\leq C_3\left\{\frac{1}{m+1}\sum_{\mu=0}^{m}\left(w_x\left(\pi,\frac{\pi}{\mu+1}\right)\right)^2\right\}^{1/2}\quad (m=0,1,2,\dots)$$

holds uniformly with respect to n, where C_3 is an absolute constant and

$$w_x(\delta,\gamma)=w_x(\delta,\gamma;f)=\sup_{\substack{0<h_1\leq\delta\\ 0<h_2\leq\gamma}}\left\{\frac{1}{h_1(h_2^2+h_1)}\int_{-h_1}^{h_1}|f(x+u)-f(x)|\,du\int_{u-h_2}^{u+h_2}|f(x+v)-f(x)|\,dv\right\}^{1/2}$$

(cf. [3]).

Theorem 3. If $f\in L^r_{2\pi}$ $\left(\frac{p}{p-1}\leq r\leq p,\ p\geq 2\right)$, then, for $x\in(-\infty,\infty)$ and $q\in(0,p\rangle$, the inequality

$$H_{m,n}[f]_q(x) \leq C_4\left\{\frac{1}{(m+1)^{r/p}}\sum_{\mu=0}^{m}(\mu+1)^{r/p-1}\left(w_x\left(\frac{\pi}{\mu+1}\right)_r\right)^r\right\}^{1/r} \quad (m=0,1,\ldots)$$

holds uniformly with respect to n, where C_4 is an positive constant depending on the parameters p, r only (cf. [4], [6]).

Analogously as before, the following result can be obtained.

Corollary 3. If $f \in Lip\,\alpha$, then, for $q \in (0,p\rangle$, the relations

$$\max_x H_{m,n}[f]_q(x) = \begin{cases} O(m^{-\alpha}) & \text{when } \alpha p < 1, \\ O(m^{-\alpha}\log^{\alpha} m) & \text{when } \alpha p = 1, \\ O(m^{-1/p}) & \text{when } \alpha p > 1, \end{cases}$$

hold uniformly with respect to n.

Remark. If $f \in L^1_{2\pi}$, then the sequence $\{S_k[f]\}$ (or the series $S[f]$) is $[C,1]^a_q$-convergent almost everywhere, for $q \in (0,2\rangle$.

References

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