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ON A CLASS OF WEIGHTED FUNCTION SPACES

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1. Pseudo-differential operators. As usual we say that a C^∞-function $p(x,\xi)$ defined on $R^{2n} = R^n_x \times R^n_\xi$ is a symbol of class $S^m_{\varrho,\delta}$ $(-\infty < m < \infty,\ 0 \le \delta \le \varrho \le 1,\ \delta < 1)$ if for any multi-indices α, β there exists a constant $c_{\alpha\beta}$ such that

$$(1) \qquad |p^{(\alpha)}_{(\beta)}(x,\xi)| \le c_{\alpha\beta} \langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|}$$

holds in R^{2n}, where $p^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\alpha_\xi D^\beta_x\, p(x,\xi)$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

The pseudo-differential operator $P(x,D_x)$ with the symbol $p(x,\xi)$ is defined by

$$P(x,D_x)\,u(x) = (2\pi)^{-n} \int e^{ix\xi}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi \qquad \text{for } u \in S(R^n)$$

where $\hat{u}(\xi) = \int e^{-iy\xi} u(y)\, dy$ denotes the Fourier transform of u. This operator maps $S(R^n)$ continuously into itself and we can extend it to a continuous operator from $S'(R^n)$ into $S'(R^n)$.

For $p \in S^m_{\varrho,\delta}$ we define the semi-norms $|p|^{(m)}_l$ by

$$(2) \qquad |p|^{(m)}_l = \max_{|\alpha|+|\beta|\le l}\ \sup_{R^{2n}} \left\{ |p^{(\alpha)}_{(\beta)}(x,\xi)| \langle\xi\rangle^{-m+\varrho|\alpha|-\delta|\beta|} \right\}.$$

Then $S^m_{\varrho,\delta}$ is a Frechét space with these semi-norms.

There are several results concerning the boundedness of pseudo-differential operators in function spaces - some references may be found in [3]. We consider here function spaces of Besov- Hardy-Sobolev type $B^s_{p,q}(R^n)$ and $F^s_{p,q}(R^n)$. These spaces are a general scale of function spaces containing a lot of classical spaces, for example the Bessel-potential-spaces, Besov-spaces, local Hardy-spaces and others. For the definition and properties of these spaces see [6].

We note the following important theorem which we shall use in the sequel. The proof may be found in [3].

__Theorem:__ Let $P(x,D_x) \in S^0_{1,\delta}$ and $\delta < 1$. Then for all p,q and s with $0 < p < \infty$, $0 < q \leq \infty$ and $-\infty < s < \infty$

$$P(x,D_x) : F^s_{p,q}(R^n) \longrightarrow F^s_{p,q}(R^n)$$

is a linear and continuous operator. Moreover, there exist two real numbers l and $c > 0$, both independent of $P(x,D_x)$, such that

$$(3) \qquad \| P(x,D_x)\, u \mid F^s_{p,q} \| \leq c \ |p|^{(o)}_l \| u \mid F^s_{p,q} \|$$

holds for all functions $u \in F^s_{p,q}(R^n)$.

__Remark:__ In the case of Besov-spaces $B^s_{p,q}(R^n)$ such a result also is true. It holds

$$\| P(x,D_x) \mid L(B^s_{p,q}, B^s_{p,q}) \| \leq c \ |p|^{(o)}_l$$

if $0 < p,q \leq \infty$ and $-\infty < s < \infty$.

In the following we consider a special class of hypoelliptic pseudo-differential operators.

__Definition:__ A symbol $a(x,\xi)$ belongs to the class $S(1,\delta;m',m)$ if there exist a constant $R \geqslant 0$ and real numbers δ, m', m with $0 \leq \delta < 1$, $0 \leq m' \leq m$, such that holds:

i) $a \in S^m_{1,\delta}$

ii) $|a^{(\alpha)}_{(\beta)}(x,\xi)| \leq c_{\alpha\beta} |a(x,\xi)| \langle\xi\rangle^{-|\alpha| + \delta|\beta|}$

for any multi-indices α, β, all $x \in R^n_x$ and all $\xi \in R^n_\xi$ with $|\xi| \geqslant R$.

iii) $c' \langle\xi\rangle^{m'} \leq |a(x,\xi)| \leq c \langle\xi\rangle^m$

with constants $c' > 0$, $c > 0$, all $x \in R^n_x$ and all $\xi \in R^n_\xi$ with $|\xi| \geqslant R$.

iv) arg $a(x,\xi)$ is a continuous and uniformlly bounded function on $R^n_x \times \{\xi : \xi \in R^n_\xi, |\xi| \geqslant R\}$.

If $a \in S(1,\delta;m',m)$, then the conditions i) - iv) allow to define a family of complex powers $\{A^z\}_{z \in \mathbb{C}}$ of the pseudo-differential operator $A(x,D_x)$. As for properties of this complex powers see [2, section 8]. There is also an expansion formula for the symbol of $A^z(x,D_x)$.

Now let us give some simple examples of symbols belonging to a class $S(1,\delta;m',m)$:

1. At first a trivial example,

$$a_1(x,\xi) = \langle\xi\rangle .$$

This symbol belongs to $S(1,0;1,1)$ and it is well-known that the complex powers of $A_1(x,D_x)$ exist.

2. Let $b_o \in S^{m'}_{1,o}$ with $c'\langle\xi\rangle^{m'} \le b_o(x,\xi) \le c\langle\xi\rangle^{m'}$, and $b_1 \in S^m_{1,o}$ with $c'\langle\xi\rangle^m \le b_1(x,\xi) \le c\langle\xi\rangle^m$, k be a natural number and $\varrho(x)$ be a real-valued function with $\sup_x |D_x^\alpha \varrho(x)| \le c_\alpha$ for all α.

If $0 \le m' \le m$ and $\delta = \frac{m-m'}{2k} < 1$, then

$$a_2(x,\xi) = b_o(x,\xi) + \varrho^{2k}(x)\, b_1(x,\xi)$$

belongs to $S(1,\delta;m',m)$.

3. Let $\varrho(x) = s + \sigma(x)$ be a real-valued function, let s be a constant and let $\sigma(x)$ be an element of $S(R^n)$. Then

$$a_3(x,\xi) = \langle\xi\rangle^{\varrho(x)}$$

belongs to $S(1,\delta;\inf\varrho(x),\sup\varrho(x))$ for any δ with $0<\delta<1$.

4. Let again $\varrho(x) = s + \sigma(x)$ and k' be an arbitrary real number. Then

$$a_4(x,\xi) = \langle\xi\rangle^{\varrho(x)}(1+\log\langle\xi\rangle^2)^{k'/2}$$

belongs to $S(1,\delta;m',m)$ with $0<\delta<1$, $m' < \inf\varrho(x)$ and $m > \sup\varrho(x)$.

2. Function spaces. The basic spaces for the following definition are the spaces $B^s_{p,q}$ and $F^s_{p,q}$ from [6]. Here we restrict ourselve to the case $F^s_{p,q}$, but all results are also true in the case of $B^s_{p,q}$.

Definition: Let $0<p<\infty$, $0<q\le\infty$, s,t,r be arbitrary real numbers and $a(x,\xi)$ a symbol belonging to $S(1,\delta;m',m)$.

$$F^{s,t,r}_{p,q}(R^n,a) = \{u:\ u\in S'(R^n) \text{ and } \|u \mid F^{s,t,r}_{p,q}(R^n,a)\| < \infty\}$$

$$\|u \mid F^{s,t,r}_{p,q}(R^n,a)\| = \|A^s(x,D_x)\,u \mid F^t_{p,q}\| + \|u \mid F^r_{p,q}\| . \tag{4}$$

Remarks: 1. The definition makes sense because the pseudo-differential operator $A^s(x,D_x)$ belongs to $S^{sm}_{1,\delta}$ if $s\ge 0$ ($S^{sm'}_{1,\delta}$ if $s<0$) and therefore it is a linear and continuous operator from $S'(R^n)$

into $S'(R^n)$. The second term in (4) is necessary in the proof that $F^{s,t,r}_{p,q}(R^n,a)$ is a quasi-Banach space.

2. If we have $s \in \mathbb{C}$, the definition is also possible. Let $s \in \mathbb{C}$ then holds

$$F^{s,t,r}_{p,q}(R^n,a) = F^{\mathrm{Re}\, s,t,r}_{p,q}(R^n,a)$$

that means, that both the operators $A^s(x,D_x)$ and $A^{\mathrm{Re}\, s}(x,D_x)$ define equivalent quasi-norms in (4).

3. In the trivial case $a_1(x,\xi) = \langle\xi\rangle$ we get by the definition the usual scale of spaces $F^{s'}_{p,q}(R^n)$ with $s' = \max(s+t,r)$.

4. If we have the spaces $B^t_{p,q}$ instead of $F^t_{p,q}$ with $1 < p < \infty$, $1 \le q \le \infty$ and the symbol $a_3(x,\xi) = \langle\xi\rangle^{\varrho(x)}$ - see example 3 - then we can get spaces of Besov-type with variable order of differentiation $B^{p,q}_{\varrho}(R^n)$ - introduced by Beauzamy [1].

5. For the symbol $a_4(x,\xi) = \langle\xi\rangle^{\varrho(x)}(1+\log\langle\xi\rangle^2)^{k'/2}$ - see example 4 - the space $F^{1,0,m'-1}_{2,2}(R^n,a_4)$ coincides with the space $H^{\varrho,k'}$ introduced by Unterberg [7].

In these cases we can find equivalent norms in [6], [1] and [7].

Now we describe some properties of the spaces $F^{s,t,r}_{p,q}(R^n,a)$. As for proofs and details see [4].

$F^{s,t,r}_{p,q}(R^n,a)$ is a quasi-Banach space if $-\infty < s,t,r < \infty$, $0 < p < \infty$ and $0 < q \le \infty$ (Banach space if $1 \le p < \infty$, $1 \le q \le \infty$).

If $-\infty < s,t,r < \infty$, $0 < p < \infty$ and $0 < q < \infty$, then $S(R^n)$ is dense in $F^{s,t,r}_{p,q}(R^n,a)$.

If $s \geq 0$ and $sm+t \le r$ (respectively $sm'+t \le r$ if $s < 0$). Then

$$F^{s,t,r}_{p,q}(R^n,a) = F^r_{p,q}(R^n) .$$

On the other hand, let $s \geq 0$ and $r,r' \le sm'+t$ ($r,r' \le sm+t$ if $s < 0$), then we have

$$F^{s,t,r}_{p,q}(R^n,a) = F^{s,t,r'}_{p,q}(R^n,a) .$$

In the first case we may say, that r is dominating and we get the usual scale of function spaces $F^r_{p,q}$. In the second case the exact values of the parameters r and r' are not essential; without loss of generality we may assume that $r = sm'+t$ (if $s \geq 0$). This will be the most interesting case for us, because here the pseudo-differential operator dominates. And at last, if $sm'+t < r < sm+t$ $(s \geq 0)$ the space $F^{s,t,r}_{p,q}(R^n,a)$ is the intersection of a space with dominating parameter r and of a space where the pseudo-differential operator dominates.

It is possible to introduce a partial ordering relation in the symbol class $S(1,\delta;m',m)$ in a natural way. This partial ordering has such properties, that we can get from it imbedding properties of the spaces $F^{s,t,r}_{p,q}(R^n,a)$. For example, two symbols which are equivalent in the sense of this partial ordering relation define the same spaces.

In [6, section 2.4] Triebel described a method of complex interpolation which is available for some special quasi-Banach spaces of functions. By this method it is also possible to get interpolation theorems for the spaces $F^{s,t,r}_{p,q}(R^n,a)$. But by the existence of three parameters s,t and r we must consider several different cases and cannot give the results here as one theorem. We only remark, that in all cases the interpolation space is either again a space of the type $F^{s,t,r}_{p,q}(R^n,a)$ or the intersection of two spaces of this type.

<u>3. Weighted function spaces.</u> We consider symbols of type $a_2(x,\xi)$ - see example 2. Let

$$a'_2(x,\xi) = \langle\xi\rangle^{m'} + \rho^{2k}(x)\,\langle\xi\rangle^{m} \tag{5}$$

where $\rho(x)$ is a real-valued function as in example 2 and m',m,k are positive integers with $0 \leq m - m' < 2k$.

We recall that

$$F^{l}_{p,2}(R^n) = W^{l}_{p}(R^n) \quad \text{if} \quad l = 0,1,2,\ \ldots \text{ and } \quad 1 < p < \infty .$$

<u>Theorem:</u> Let $a'_2(x,\xi)$ be defined by (5) and $t = r = 0$, s be a natural number, s or m are even and $1 < p < \infty$, $q = 2$.

Then

$$\| u \mid W^{sm'}_{p} \| + \sum_{|\gamma| = sm} \| \rho^{2ks}(x)\, D^{\gamma}_{x} u \mid L_p \|$$

is an equivalent norm in the space $F^{s,o,o}_{p,2}(R^n,a'_2)$.

<u>Remarks:</u> 1. The theorem is also true if we take the symbol $a_2(x,\xi)$ from example 2, where again m',m,k,s must be natural numbers and s or m is a even number. For the same parameters m',m and k, $a'_2(x,\xi)$ and $a_2(x,\xi)$ are equivalent in the sense of the partial ordering relation in $S(1,\delta;m',m)$ and define the same spaces.

2. In this special case we can characterize the space $F^{s,o,o}_{p,2}(R^n,a_2)$ by weighted derivatives in the usual sense. This is also possible, if t,r are arbitrary real numbers and $0 < p < \infty$, $0 < q \leq \infty$. These parameters were fixed in the theorem only for simplicity.

3. The proof of the theorem contains estimates for the "mixed" derivatives; for example

$$\|\varrho^{j}(x)\, D_x^{\alpha} u \mid L_p\| \leq c\, (\|u|W_p^{sm'}\| + \sum_{|\gamma| = sm} \|\varrho^{2ks}(x)\, D_x^{\gamma} u \mid L_p\|$$

if $1 < p < \infty$ and $1 \leq j \leq 2ks$, j is a real number and $|\alpha| \leq sm' + \frac{m-m'}{2k} \cdot j$.

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