КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 534–543

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
Sofia, 1984, p.534-543

EXTREMAL PROBLEMS IN CLASSES OF SUBHARMONIC FUNCTIONS AND THEIR APPLICATIONS

B. Ja. Levin

This paper deals with the theory of majorants of the special classes of functions which are subharmonic in the whole plane $\mathbb{C}$ and bounded in a certain sense on the real axis. The paper will be restricted to a brief review of these theorems, without any proofs supplied.* The theorems which are obtained as applications of the principal theorems contain general outlines of proofs.

<u>Classes K_φ .</u> We shall denote by $\varphi(x) \geq 0$ an arbitrary function upon $\mathbb{R}$, the values $\varphi(x) = +\infty$ not being excluded, and by K_φ any class of functions $g(z)$, subharmonic in $\mathbb{C}$, that satisfies the following conditions:

a) All functions of K_φ satisfy the following inequality:

$$g(x) \leq \varphi(x);$$

b) If $g(z) \in K_\varphi$ and the function $g_c(z)$ which is a result of balayage from a certain circle C satisfies the condition a), then $g_c(z) \in K_\varphi$;

c) If $g_1(z), g_2(z) \in K_\varphi$, then $g(z) = max(g_1(z), g_2(z)) \in K_\varphi$;

d) $0 \in K_\varphi$;

e) There exists a nonconstant function $g(z) \in K_\varphi$.

---

* A reprint [1] is now about to appear, where these theorems will be presented with detailed proofs.

In what follows we shall consider, without specifying it, only the symmetric classes K_φ, that is such classes K_φ that $g(z) \in K_\varphi$ implies $g(\bar{z}) \in K_\varphi$.

A function $g(z)$, subharmonic in $\mathbb{C}$, satisfying the condition

$$\sigma = \overline{\lim_{|z| \to \infty}} \frac{g(z)}{|z|} < \infty$$

will be called a function of finite degree σ. The class of subharmonic functions which contains all functions satisfying the condition a) and having finite degrees $\leq \sigma$ will be denoted by K_φ^σ. By $K_{\varphi,\psi}^\gamma$ we shall denote the class of all functions from K_φ^0 satisfying the condition

$$\overline{\lim_{y \to +\infty}} \frac{g(iy)}{\psi(|y|)} \leq \gamma < \infty,$$

where $\psi(|y|)$ is some monotonically increasing function, and put $K_{\varphi,\psi} = \bigcup\limits_{\gamma > 0} K^\gamma_{\varphi,\psi}$

<u>Majorant of a class K_φ.</u> Let us put

$$\hat{g}(z, K_\varphi) = \sup_{g \in K_\varphi} \{g(z)\}$$

and identify as the majorant of a class K_φ the function $v(z, K_\varphi)$ which is obtained from $\hat{g}(z, K_\varphi)$ by regularization:

$$v(z, K_\varphi) = \lim_{\delta \downarrow 0} \big(\sup_{|\zeta - z| < \delta} \hat{g}(\zeta, K_\varphi)\big).$$

Sometimes we shall write just $v(z)$.

<u>Theorem 1.</u> The majorant $v(z)$ of a class K_φ is either equal to $+\infty$ everywhere in $\mathbb{C}$ or is finite everywhere. In the latter case, $v(z)$ is a harmonic function in $\mathbb{C}_+$ and $\mathbb{C}_-$ and upon those intervals of the real axis $\mathbb{R}$, where $v(x) < \varphi(x)$. Besides, the finite majorant is representable as

$$v(z) = \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{v(t)}{|t - z|^2}\, dt + \sigma |y|,$$

where $\sigma \geq 0$.

It follows from the latter formula that if the majorant of a class K_φ is finite, then K_φ is a part of some class K_φ^σ and the majorant has a finite degree.

The complex majorant of a class K_φ is said to be a function $w(z)=w(z, K_\varphi)$, holomorphic in $\mathbb{C}_+$

$$w(z) = u(z) + i v(z) \tag{1}$$

whose imaginary part $v(z)$ is the majorant of K_φ.

Theorem 2. If a function $\varphi(x) \geq 0$ satisfies, for all $x \in \mathbb{R}$, the conditions:

1. The set E of points, where $\varphi(x)$ is finite, must be closed;
2. $\varphi(x)$ is lower semi-continuous upon E;
3. Every point $x_0 \in E$ is regular for a Lebesgue set $E_\varepsilon = \{x : \varphi(x) \leq \varphi(x_0) + \varepsilon\}$, $\varepsilon > 0$, (i.e. a regular point of the domain $\mathbb{C} \setminus E_\varepsilon$);

and if a majorant $v(z)$ of a certain class K_φ is finite, then the corresponding complex majorant (1) is continuously extendable onto $\mathbb{R}$.

The S-curve is by definition a continuous curve $u = u(x)$, $v = v(x) \geq 0$ ($-\infty < x < \infty$), if it consists of α) a graph of a function $v = \theta(u)$ ($-\infty \leq a < u < b \leq +\infty$) with only points of first kind discontinuity and at these points of discontinuity $\theta(u_j) = \min(\theta(u_j+0), \theta(u_j-0))$ and β) segments $h_j = \{w : u = u_j =$ const, $a_j \leq v \leq b_j\}$, $j = 1, 2, \ldots$, where $a_j = \theta(u_j)$ and $b_j \geq \max(\theta(u_j+0), \theta(u_j-0))$.

The Ω_S-domain is by definition a domain bounded by an S-curve in such a way that at every point $w = u + iv \in \Omega_S$, $v > \theta(u)$ (Fig. 1).

The following theorem is central.

Theorem 3. For function $w = w(z)$, holomorphic in $\mathbb{C}_+$ to be a complex majorant of a certain class K_φ with $\varphi(x)$ satisfying conditions 1), 2), 3) of Theorem 2 it is necessary and sufficient that w should be a conformal mapping of the half-plane $\mathbb{C}_+$ onto a certain Ω_S-domain.

This theorem provides a method to solve some extremal problems in classes of subharmonic functions by reducing them to finding some special conformal mappings.

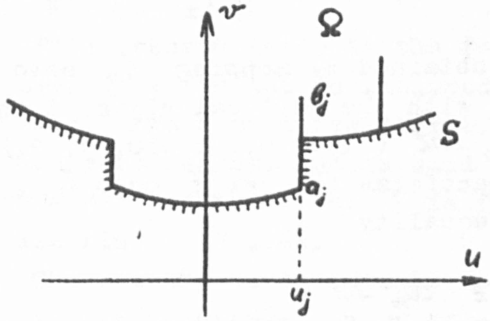

Fig. 1

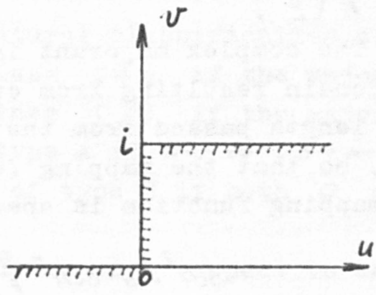

Fig. 2

<u>Problem 1*.</u> Let $\varphi(x) = 0$ for $x \leq 0$ and $\varphi(x) = 1$ for $x > 0$. Find the majorant $v(z)$ in the class K_{φ}^{σ} of subharmonic functions for a given $\sigma > 0$.

Theorem 3 reduces the problem to finding a mapping of the half-plane C_+ onto the domain shown on Fig. 2, normalized in a certain way at infinity. Having found this mapping, we obtain

$$v(z) = \frac{1}{\pi} \operatorname{Im}\left\{ \ln\left(z_1 + \sqrt{z_1^2 - a^2}\right) + \sigma\pi\sqrt{z_1^2 - a^2} \right\},$$

where $a = (\pi\sigma)^{-1}$ and $z_1 = z - a$.

<u>Problem 2.</u> Find the majorant of K_{φ}^{σ} for $\varphi(x) = 0$ ($|x| \leq 1$) and $\varphi(x) = A > 0$ for $|x| > 1$.

The complex majorant is given by the Kristoffel-Schwartz formula

$$w(z) = \frac{\sigma}{k} \int_0^z \sqrt{\frac{1 - k^2\zeta^2}{1 - \zeta^2}}\, d\zeta \qquad (0 < k < 1)$$

The set E , where $v(x) = \varphi(x)$, consists apparently of the intervals $|x| \leq 1$ and $|x| \geq \frac{1}{k}$, k being specified by equation

* At Cornell University Colloquium ([2], Problem 3) the problem was formulated on a sharp upper estimate for $g(z) = \ln|f(z)|$, $f(z)$ being an entire function of a finite degree $\leq \sigma$, $g(x) \leq 0$ for $x < 0$ and $g(x) \leq 1$ for $x \geq 0$. Here we supply the solution of this problem but in a wider class of functions. On this problem, see also Ref. 3.

Problem 3. Find the majorant of K_{φ}^{σ} for $\varphi(x) = 0$ upon the set $E=\{x: |x-n+\frac{1}{2}| \leq d < \frac{1}{2}, \quad \pm n = 0, 1, 2, \ldots\}$ and $\varphi(x)=+\infty$ upon $\mathbb{R}\setminus E$.

The complex majorant is readily obtained by mapping $\mathbb{C}_+$ onto the domain resulting from cutting $\mathbb{C}_+$ with the vertical slits of the same length passed from the points $x_n = n\ell$ ($\ell > 0$, $n = \pm 0, 1, 2, \ldots$), so that the mapping function transforms the set E onto $\mathbb{R}$. The mapping function is specified by equality

$$\sin^2 \pi d \cdot \sin^2 \frac{\pi}{\ell} w = \sin^2 \pi z - \cos^2 \pi d . \tag{2}$$

From this solution of Problem 3 immediately follows

Theorem 4. [4] If a subharmonic function $g(z)$ satisfies the conditions of Problem 3, then

$$\max_{x\in\mathbb{R}} g(x) \leq \frac{\sigma}{\pi} \ln \operatorname{ctg} \frac{\pi d}{2} . \tag{3}$$

Corollary. If an entire function $f(z)$ of an exponential type $\leq \sigma$ satisfies inequality $|f(x)| \leq 1$ on the set E of Problem 3, then it satisfies the estimate

$$|f(x)| \leq \left(\operatorname{ctg} \frac{\pi d}{2} \right)^{\frac{\sigma}{\pi}} \tag{4}$$

for all $x \in \mathbb{R}$. The most important mappings of those discussed in Theorem 3 are the E -correct mappings introduced in Refs. 5 and 6.

The E -Correct Mapping is a conformal mapping of $\mathbb{C}_+$ onto a domain Ω which is a domain $\operatorname{Im} z>0; -\infty \leq a < \operatorname{Re} z < b \leq +\infty$, cut by a finite or countable set of the vertical slits beginning on $\mathbb{R}$ and having a finite length. It is also required that the given closed set E should be mapped onto the "base of the domain" $\operatorname{Im} z = 0$ and every interval from $\mathbb{R}\setminus E$ should be mapped onto some vertical slit.

Theorem 5. [4 , 5 , 6]. If a set E consists of regular points of the boundary of the domain $\mathbb{C}\setminus E$, then the corresponding

E -correct mapping $w = w(z)$ does exist and is defined by the set E up to a constant positive multiplier λ and a real shift μ, i.e. $w = \lambda w_0(z) + \mu$.

This theorem leads to the following natural classification of the sets E : the set is included in the class (α) if the majorant $v(z)$ has a positive degree, in the class (β) if the majorant has zero degree, and is said to be of type A if $a = -\infty$, $b = \infty$, of type B if either a or b is finite and of type C if both a and b are finite [5, 6].

The set E is said to be relatively dense with respect to measure, if every interval in $\mathbb{R}$ of a length l contains a portion of of a measure $\geqslant \delta > 0$.

Theorem 6. If E is relatively dense with respect to measure, then the E -correct mapping belongs to the class (α) and its imaginary part is bounded upon $\mathbb{R}$, i.e.

$$v(x) \leq c(l, \delta)\sigma,$$

where σ is the degree of $v(z)$ *.

The relationship between E -correct mappings and the spectral theory of differential operators was first established by V.A. Marchenko and I.V. Ostrovskii [8, 9, 10] and studied in Refs. 11 and 12. We consider a generalized differential equation

$$y'(x) + \lambda^2 \int_0^x y(t)\, d\sigma(t) = const, \tag{5}$$

where $\sigma(t)$ is a non-decreasing function; M.G. Krein called it the string equation **. The string is called π-periodic if $\sigma(x+\pi) - \sigma(x) =$ const. Note that in case of sufficiently smooth function $\sigma(x)$,

* In other terms, this theorem is presented in Refs. 7, 5 and 6. The author thinks that the right-hand side of Eq. (3) gives the exact $c(l, \delta)$ value (for $l = 1$, $\delta = d$) which is attained only for the imaginary part of the function w defined by equality (2). A related question was considered by E.V. Glezer and A.A. Goldberg.

** A spectral theory of the string was constructed by M.G. Krein and was published in DAN USSR. A detailed presentation of this theory and a list of sources may be found in Ref. 12.

the equation of a periodic string is reducible by usual substitutions to the Hill equation *.

Theorem 7. A set E is a spectrum of some string if and only if the corresponding E -correct mapping $w=u(z)+iv(z)$ transforms $\mathbb{C}_+$ onto "a comb", that is onto domain $v(z)>0$, $|u(z)|<n\pi$, n being an integer or $+\infty$, with the cuts $h_k=\{u=k\pi,\ 0\le v\le \le \ell_k<\infty$, $\ell_k=\ell_{-k}$ and $k = 0, 1, 2, \ldots$ Let us outline the proof of the theorem.

Lemma. If a function $w(z)=u(z)+iv(z)$ is holomorphic in $\mathbb{C}_+$ and is continuously extendable to $\mathbb{R}$, $v(z)\ge 0$ and $u(x)$ is a non-decreasing function $\mathbb{R}$, then $w(z)$ is univalent in $\mathbb{C}_+$.

The proof consists in application of the argument principle to a function $w(z)+\varepsilon z$; $\varepsilon>0$, in a semi-circle $\Im m\, z>0$, $|z|<R$, followed by a passage to the limit for $R\to\infty$ and $\varepsilon\to 0$.

To prove the necessity in Theorem 7, we are to construct, as usual, the Flouquet solutions of Eq. (5):

$$y(x+\pi,\lambda)=\rho(\lambda)\,y(x,\lambda) \tag{6}$$

the multiplicators $\rho(\lambda)$ satisfying the Ljapunov equation

$$\rho^2-2A(\lambda)\rho+1=0 \qquad (\rho=\rho(\lambda),\ \lambda\in\mathbb{C}) \tag{7}$$

Here $2A(\lambda)=c(\pi,\lambda)+s'(\pi,\lambda)$, $c(x,\lambda)$ and $s(x,\lambda)$ are the solutions of Eq. (5) with $c(0,\lambda)=s'(0,\lambda)=1$ and $c'(0,\lambda)=s(0,\lambda)=0$. The spectrum of the string (otherwise called "stability zones") obviously coincides with the set of the λ 's for which $|\rho(\lambda)| = 1$. It is easy to see that the spectrum of the string is real. Indeed, multiplication of (5) by $\bar{y}(t,\lambda)$ satisfying (6) and integration between 0 and 1 result in the following:

$$(|\rho(\lambda)|^2-1)\,y'(0,\lambda)y(0,\lambda)-\int_0^1|y'(t,\lambda)|^2dt+\lambda^2\int_0^1|y(t,\lambda)|^2d\sigma(t)=0 \tag{8}$$

Hence $|\rho(\lambda)| = 1$ for $\lambda^2>0$ only.

---

* The direct and inverse spectral problems for a periodic string were investigated by M.G. Krein [13].

Choose for $\lambda \in \mathbb{C}_+$ such a multiplier $\rho(\lambda)$ that $|\rho(\lambda)| > 1$ in $\mathbb{C}_+$. $\rho(\lambda)$ is a real function for those λ which do not belong to a spectrum. Hence, the function $\theta(\lambda) = i \ln \rho(\lambda)$ maps $\mathbb{C}_+$ into $\mathbb{C}_+$ and, as λ changes along $\mathbb{R}$, the function $\theta(\lambda)$ monotonically runs over "a comb". By the Lemma, $\theta(\lambda)$ is an E -correct mapping (E being a spectrum) of $\mathbb{C}_+$ onto "a comb". It is obvious that $A(\lambda) = \cos\theta(\lambda)$.

The sufficiency follows from a theorem of M.G. Krein [13], if note that $A(\sqrt{\lambda})$ is an entire function of a zero genus and $A^2(\sqrt{\lambda}) - 1 = 0$ has only real roots *.

<u>Some applications of subharmonic majorants to quasi-analytic classes.</u> The closure E of a subset of $\mathbb{R}$, where the function

$$\psi(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx \tag{9}$$

$f(x) \in L(\mathbb{R})$ does not vanish, will be called <u>the spectrum</u> of a function $f(x)$.

<u>Theorem 8.</u> If the spectrum of a function $f(x)$ is a set of the class (β) and $f(x) = 0$ upon some interval, then $f(x) = 0$ a.e.

<u>Proof.</u> Without loss of generality, we may assume that $f(x) = 0$ for $|x| < \delta$ ($\delta > 0$). Put

$$\Phi_+(\lambda) = \int_0^{\infty} f(x) e^{i\lambda x} dx, \ (\lambda \in \mathbb{C}_+), \ \Phi_-(\lambda) = -\int_{-\infty}^{0} f(x) e^{i\lambda x} dx \quad (\lambda \in \mathbb{C}_-)$$

For $\lambda \in \mathbb{R} \setminus E$ the boundary values coincide, i.e. $\Phi_+(\lambda) = \Phi_-(\lambda)$ and by the theorem on removing of singularities we obtain the

* Recently M.G. Krein informed me that he had proved a more general fact: a mapping of any "comb" (without the requirement $h_k = h_{-k}$) onto $\mathbb{C}$ transforms the "comb" base into a spectrum of some canonical periodic system

$$\frac{dy}{dt} = \lambda J H(t) y; \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and all the spectra of such systems are obtainable likewise. The same fact was independently proved by I.V. Mikhaylova.

function $\Phi(\lambda)$ which is holomorphic in $\mathbb{C}\setminus E$, is continuous in $\mathbb{C}$ and satisfies the estimate

$$(10) \qquad |\Phi(\lambda)| < Me^{-\delta|\nu|} \qquad (\lambda=\mu+i\nu\in\mathbb{C})$$

The E -correct mapping $w_E(z)=u_E(z)+iv(z)$ is extendable to $\mathbb{C}_-$ through $\mathbb{R}\setminus E$ in such a way, that $v_E(\bar\lambda)=v_E(\lambda)$, $v_E(\mu)=0$ for $\mu\in E$, $v(z)>0$ for $z\in\mathbb{C}\setminus E$ and

$$(11) \qquad \lim_{|\lambda|\to\infty}\frac{v_E(\lambda)}{|\lambda|}=0.$$

For any fixed $N > 0$, the function $K_N(\lambda)=\ln|\Phi(\lambda)|+ + N\ln v_E(\lambda)$ is bounded, as suggested by (10) and (11), from above by some constant C_N in the angles $|\arg\lambda\pm\frac{\pi}{2}|\le\frac{\pi}{2}-\varrho$ $(0<\varrho<\frac{\pi}{2})$. Besides, by applying the maximum principle to the function $K_N(\lambda)+N\ln v_E(\lambda)-\varepsilon|\lambda|^{\rho}\cos(\rho\varphi)$; $\varphi=\arg\lambda$, $1<\rho<\frac{\pi}{\varrho}$, in the domain $(\lambda:|\arg\lambda|<\varrho)\setminus E$ and similarly in the domain $(\lambda:|\arg\lambda-\pi|<\varrho)\setminus E$, after passage to the limit for $\varepsilon\downarrow 0$, we obtain $K_N(\lambda)<\max(C_N, M)$. Finally, by applying the Phragmen-Lindelöf principle in $\mathbb{C}\setminus E$, we obtain $K_N(\lambda)\le\ln M$ for any $N>0$, and it follows that $\Phi(\lambda)\equiv 0$, which means that $f(x)=0$ a.e.

Theorem 9. If E is the spectrum of a string, then the class of the functions from $L(\mathbb{R})$ whose spectra belong to E is I-quasianalytical if and only if E belongs to the class (β).

References

1. B.Ja.Levin. Extremal problems in classes of subharmonic functions and their applications (in Russian). Preprint of Institute for Low Temperature Physics and Engineering, UkrSSR Academy of Sciences, Part I, No. 18-84; Part II, No. 19-84.
2. Research Problems in Function Theory. Bull. of Amer. Math. Soc. V. 68, No. 1, p. 21-24, 1969.

3. A.A.Goldberg and B.Ja.Levin. On entire functions bounded on the real axis (in Russ.). DAN SSSR, 1964, V. 157, No. 1, 19-21.
4. B.Ja.Levin. Subharmonic majorants and applications. Abstracts of the All-Union Conference on the Theory of Complex Variable Functions. Kharkov, 1971, 117-120.
5. N.I.Akhiezer and B.Ja.Levin. Inequalities for derivatives analogous to the Bernstein inequality (in Russ.). DAN SSSR, 1958, V. 96, No. 2, 256-258.
6. N.I.Akhiezer and B.Ja.Levin. Generalization of the S.N. Bernstein inequality. Studies into present problems of the theory of functions of a complex variable (in Russ.). Fizmatgiz, Moscow, 1960, 131-165.
7. A.C.Shaeffer. Entire functions and trigonometric polynomials. Duke Math. Journ., 20 (1953) 77-88.
8. V.A.Marchenko and I.V.Ostrovskii. Characterization of the Hill operator spectrum. Matem. Sbornik, 1975, V. 97, No. 4, 540-606. (in Russ.).
9. V.A.Marchenko and I.V.Ostrovskii. Approximation of periodic potentials by finite-zone potentials (in Russ.). Vest. Khark. Gos. Univ., No. 205, Prikladnaya matematika i mekhanika, vyp. 45, Kharkov, 1980, 4-40.
10. V.A.Marchenko. Sturm-Liouville operators and their applications (in Russ.). Naukova Dumka, Kiev, 1977.
11. B.M.Levitan. Approximation of infinite-zone potentials by finite-zone potentials (in Russ.). Izvestiya AN SSSR, ser. matem., 1982, V. 46, No. 1, 56-87.
12. L.A.Pastur and V.A.Tkachenko. On the spectral theory of the one-dimensional Schrödinger operator with a limit-periodic potential. DAN SSSR, to be published.
13. H.Dym and H.P.McKean. Haussian processes function theory in the inverse spectral problem. Academic Press, 1976.
14. M.G.Krein. On inverse problems of the theory of filters and -zones of stability (in Russ.). DAN SSSR, 1953, V. 93, No. 5, 767-770.

Institute for Low Temperature Physics and Engineering,
UkrSSR Academy of Sciences,
47 Lenin Prospect, Kharkov 310164,
USSR