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AN EXISTENCE THEOREM FOR MINIMAL LINEAR PROJECTIONS

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0. Introduction. Let X be a normed linear space over K ($K=\mathbb{R}$ or $K=\mathbb{C}$) and let Y be its closed linear subspace. A linear operator $P:X\rightarrow Y$ is called a projection if P is continuous and $Py=y$ for each $y\in Y$. The set of all projections $P:X\rightarrow Y$ is denoted by $P(X,Y)$. (For basic properties see e.g. [1], [3], [5].)

We recall that an operator $P_0\in P(X,Y)$ is said to be minimal if $\|P_0\|\leq\|P\|$ for all $P\in P(X,Y)$. If $\|I-P_0\|\leq\|I-P\|$ for each $P\in P(X,Y)$ then we say that P_0 is co-minimal. It is very important to ask whether minimal and co-minimal projection exist, because the following inequality holds:

$$\|x-Px\|\leq\|I-P\|\operatorname{dist}(x,Y)\quad (x\in X, P\in P(X,Y)).$$

In the case when Y is a dual space ($Y=Z^*$ for some normed linear space Z) this problem was solved by E.W.Cheney and P.D.Morris (see [3], theorem 3.4, p.34). They proved

0.1. Theorem. Assume that X is a normed space and $Y\subset X$ is its closed linear subspace. Assume furthermore that Y is a dual space and the set $P(X,Y)$ is non-void. Then the set $P(X,Y)$ is proximinal in $L(X,Y)$.

In this note which is a short version of a more detailed article [4] we shall give a generalization of Theorem 0.1.

1. The theorem and examples.

1.1. Theorem. Let X be a normed space and let Y be a Banach space both X and Y over the same field K. Assume that there exists a set $W \subset Y^*$ which satisfies the following conditions

(1) $\|f\| = 1$, $f \in W$;

(2) $\sup_{f \in W} |f(y)| = 1$, as $y \in Y$ and $\|y\| = 1$;

(3) The space Y is quasi-complete in the topology $*$ which is induced on the space Y by the set W: a net $y_\beta \to y$ iff $f(y_\beta) \to f(y)$ for all $f \in W$.

Let us introduce a topology τ in $L(X,Y)$ in the following way: a net $L_\beta \to L$ iff $f(L_\beta x) \to f(Lx)$ for all $f \in W, x \in X$. Then every non-void, τ-closed set $U \subset L(X,Y)$ is proximinal in $L(X,Y)$.

Proof. At first we shall prove that $K_{L(X,Y)}(0,1)$ is τ-compact. Let us define the function $F: K_{L(X,Y)}(0,1) \to \mathbb{P}_{f \in W} (Y_f)$ where $Y_f = K_{X^*}(0,1)$ for all $f \in W$, by the formula: $F(L) = (f \circ L)_{f \in W}$. We will show that the function F is a homeomorphism of $K_{L(X,Y)}(0,1)$ onto its image in $\mathbb{P}_{f \in W} (Y_f)$ (in the space $K_{X^*}(0,1)$ we have $*$-weak topology and we consider the space $\mathbb{P}_{f \in W} (Y_f)$ with the Tychonoff topology). To this end suppose $F(L_1) = F(L_2)$ for $L_1, L_2 \in K_{L(X,Y)}(0,1)$ and fix $x \in X$. Then we have $f(L_1 x) = f(L_2 x)$ for every $f \in W$ and by assumption (2) we obtain $L_1x=L_2x$. Moreover the functions F and F^{-1} are continuous, because the following conditions are equivalent:

(i) $L_\beta \xrightarrow[\tau]{} L$ $(L_\beta, L \in K_{L(X,Y)}(0,1))$;

(ii) $f(L_\beta x) \to f(Lx)$ for every $x\in X$ and $f\in W$;

(iii) $f\circ L_\beta \to f\,L$ in the $*$-weak topology for every $f\in W$;

(iv) $F(L_\beta) \to F(L)$ $(L_\beta, L\in K_{L(X,Y)}(0,1))$.

By the theorems of Tychonoff and Alaoglu, in order to prove that $K_{L(X,Y)}(0,1)$ is τ-compact, it is sufficient to show that $F(K_{L(X,Y)}(0,1))$ is τ-closed in $\prod_{f\in W}(Y_f)$. Suppose $F(L_\beta)\to G\in \prod_{f\in W}(Y_f)$. Then $f\circ L_\beta \to G_f \in K_{X^*}(0,1)$ for every $f\in W$. Let us fix $x\in X$. Therefore $f(L_\beta x)\to G_f(x)$ for all $f\in W$, whence it is easily seen that the net $(L_\beta x)$ satisfies the Cauchy condition with respect to the topology $*$. We note that $\sup\|L_\beta x\| \leq \|x\|$. Hence by assumptions (2) and (3) there exists exactly one $y\in Y$ such that $L_\beta x\to y$. Let us define a map $L:X\to Y$ by the formula $Lx=\lim_\beta (L_\beta x)$ $(x\in X)$. We claim that $L\in K_{L(X,Y)}(0,1)$. Indeed, it is easily seen that L is a linear map. To prove that $\|L\|\leq 1$, fix $x\in X$ and $\varepsilon>0$. By assumption (2) we have $\|Lx\|\leq |f(Lx)|+\varepsilon$ for some $f\in W$. Then we obtain: $\|Lx\|\leq$ $\leq |f(Lx)|+\varepsilon\leq |f(Lx)-f(L_\beta x)|+|f(L_\beta x)|+\varepsilon\leq 1+2\varepsilon$ for $\beta\geq\beta_0(f)$ which completes the proof. We can show in the same way that an arbitrary closed ball in $L(X,Y)$ is τ-compact.

Now let us fix $L\in L(X,Y)$. Suppose U is a non-void, τ-closed subset of $L(X,Y)$. We have to prove that the set $F_U(L)=\{M\in U:\ \|M-L\|=\mathrm{dist}(L,U)\}$ is non-void. For each $n\in N$, define the sets B_n and A_n as follows: $B_n=K_{L(X,Y)}\left(L,\mathrm{dist}(L,U)+\frac{1}{n}\right)$, $A_n=B_n\cap U$. We note that A_n is non-void, $A_{n+1}\subset A_n$ and A_n is τ-closed for $n=1,2\ldots$. We also have $\bigcap_{i=1}^{n} A_i=A_n\neq\emptyset$ $(n\in N)$. Then $\bigcap_{i=1}^{\infty} A_i\neq\emptyset$, because A_1 is τ-compact. For $P\in\bigcap_{i=1}^{\infty} A_i$ we have $P\in U$ and $\|P-L\|\leq$ $\leq\mathrm{dist}(L,U)+\frac{1}{n}$ for all $n\in N$, whence $P\in P_U(L)$. This completes the proof.

1.2. Corollary. Assume that $Y\subset X$ and X,Y,W are such as in Theorem 1.1. Assume furthermore that the set $P(X,Y)$ of all continuous linear projections from X onto Y is non-void. Then $P(X,Y)$ is τ-closed, so it is proximinal in $L(X,Y)$.

In particular, by Corollary 1.2 we get Theorem 0.1, since we have the following

1.3. Remark. Assume that the Banach space Y is dual to some normed space Z. Then the set $W=\{\hat{z}:z\in Z,\|z\|=1\}$, where $\hat{z}:Z^*\ni f\to f(z)\in K$ satisfies the conditions (1), (2) and (3) of Theorem 1.1.

Proof. It is easily seen that the set W fulfils (1) and (2). To prove that W satisfies (3) let us suppose a net (y_β) is a Cauchy net with respect to the topology $*$ and $\sup\|y_\beta\|<\infty$. Then we have $\hat{z}(y_\beta)=y_\beta(z)\to g_z\in K$ for every $z\in Z$. Let us define a map $y:Z\to K$ by the formula: $y(z)=g_z$ $(z\in Z)$. We claim that $y\in Z^*$. Indeed, if $z\in Z$, $\|z\|=1$ and $\epsilon>0$ then we have $|y(z)|\leq|y(z)-y_\beta(z)|+|y_\beta(z)|\leq$ $\leq\epsilon+\sup\|y_\beta\|<\infty$ for $\beta\geq\beta_0(z)$. It is obvious that y is a linear map. The proof is completed.

Now we shall give two examples of Banach spaces Y which fulfil the assumptions of Theorem 1.1 (for the detailed proofs see [4]).

1.4. Example. Let X,Z be arbitrary normed spaces and $Y=L(X,Z^*)$. For $f\in Z^{**}$ and $x\in X$, define a function $\langle f,x\rangle\in Y^*$ by the formula: $\langle f,x\rangle(L)=f(Lx)$ $(L\in Y)$ and put $W=\{\langle\hat{z},x\rangle : x\in X, \|x\|=1, z\in Z, \|z\|=1\}$. Then the space Y and the set W satisfy (1), (2) and (3) of Theorem 1.1.

1.5. Example. Let T be an arbitrary set. Let Z be an arbitrary normed space and $Y=B(T,Z^*)$ (the space of all bounded functions from T into Z^* with the supremum norm). For $f\in Z^{**}$ and $t\in T$, define a function $\langle f,t\rangle\in Y^*$ by $\langle f,t\rangle(G)=f(G(t))$ $(G\in Y)$. Then the space Y and the set $W=\{\langle\hat{z},t\rangle : t\in T, z\in Z, \|z\|=1\}$ fulfil (1), (2) and (3) of Theorem 1.1.

We note that for uncountable sets T the spaces $B(T,R)$ and $B(T,\mathbb{C})$ are not dual (see [2], p.492) whence Theorem 1.1 is essentially stronger than Theorem 0.1.

References

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