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SOME REGULAR PROBLEMS OF BIVARIATE INTERPOLATION

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An interpolation problem is understood to be a set of n points Z, an n-dimensional linear space of functions V_n and a set of n values associated to the points. The solution is a function in V_n which takes the given values at the given points. On the real line, there are choices for V_n such that the interpolation problem has a unique solution for any choice of knots and any choice of values. In this case, we say that the interpolation problem is regular. In the plane, the situation becomes more complicated because the interpolation problem as posed above can be regular only in the trivial case of one point.

There are several ways to get around this problem. First, one could look for particular choices of sets of points, for example, those on a regular grid, for which the problem has a solution. This is done in the theory of finite elements, bivariate splines and with the natural lattices of Chung and Yao [1]. One could also change the type of functional which is being interpolated. Thus, instead of point evaluation functionals, one could take values of derivatives or means of values or of derivatives. These possibilities have been investigated in the theory of finite elements and splines (with fixed knot sets) and the interpolation theories of Kergin [6] and Hakopian [4]. This approach does indeed yield some interpolation problems which are regular in the above sense and we can characterize them completely. They are, however, not of practical interest except for the case of the truncated Taylor expansion.

Also possible are the Newton-Type methods of Werner [9], Gasca and Maeztu [3] and Jetter [5]. They choose a set of lines which induces the point set Z and associate to them the interpolating space $V_n(Z)$. It follows that the interpolation space is variable. However, as a consequence, the interpolation problem is always solvable.

Our approach will be to fix the set of functions from which we interpolate and then to investigate which Birkhoff interpolation problems (interpolation of the value of a function and its derivatives) have solutions for <u>some</u> set of points. Because our interpolating spaces are spaces of polynomials, this will imply that the interpolation problem is then solvable for almost all choices of points.

To be precise, the interpolating space is determined by a lower set S. This is a finite set of pairs of non-negative integers (i,k) for which $(i', k') \le (i, k)$ (that is $i' \le i$ and $k' \le k$) imply $(i', k') \in S$. The polynomial space P_S determined by S is the space of all polynomials $P(x,y)$ with

$$P(x,y) = \sum_{(i,k)\in S} a_{ik}\, x^i y^k . \qquad (1)$$

The interpolation conditions are determined by an incidence matrix $E = (e_{q,i,k})$ whose elements $e_{q,i,k}$ are zeros or ones for $q=1, \ldots, m$ and $(i,k) \in S$. The set of points of interpolation is $Z=\{z_q=(x_q,y_q) \mid q=1, \ldots, m\}$. Given Z, P_S and a set $C = \{c_{q,i,k}\}$ of real numbers, the interpolation problem is to find a polynomial $P \in P_S$ satisfying

$$\frac{\partial^{i+k}}{\partial x^i \partial y^k} P(x_q,y_q) = c_{q,i,k} \qquad (2)$$

for all q,i,k with $e_{q,i,k} = 1$. These equations form a system of linear equations for the coefficients a_{ik} of P. If (E,Z) is given, the problem (2) is solvable if and only if the determinant D(E,Z) of the system is not zero. Then the pair (E,Z) is called regular. The matrix E is regular if (E,Z) is regular for all $Z \subset R^2$.

Now D(E,Z) is a polynomial of high degree in the 2m variables x_q, y_q. Either it is identically zero or it vanishes on a set of measure zero in R^{2m}. In the first case, E is singular. In the second case, we say that E is conditionally regular. We present some new interpolation problems which are conditionally regular. For some of them, we can also determine sets of points for which (E,Z) is regular. We also characterize regular matrices completely.

Before presenting the regularity theorems, we would like to discuss the spaces of bivariate polynomials used for interpolation. The most common ones are the spaces P_n of polynomials of total degree n whose elements are

$$P(x,y) = \sum_{i+k\leq n} a_{ik}\, x^i y^k \qquad (3)$$

and the spaces $Q_{n,m}$ of polynomials of coordinate degree (n,m) whose elements are

$$P(x,y) = \sum_{\substack{i\leq n\\ k\leq m}} a_{ik}\, x^i y^k. \qquad (4)$$

Both these spaces are of the form P_S for the appropriate S. Both of them are invariant under decoupled affine transformations of the variables $x\to ax+b$, $y\to cy+d$. But only the P_n are, in addition, invariant under rotations. In fact, it is easy to see that if a finite-dimensional linear polynomial space is invariant under all regular affine transformations of the plane, then it must be P_n for some n.

The spaces P_S determined by a lower set S are characterized similarly.

<u>Lemma 1</u> Let V be a finite-dimensional linear subspace of bivariate polynomials which is invariate under any transformation of variables of the form $x\to ax + b$, $y\to cy+d$. Then $V=P_S$ for some lower set S.

<u>Proof</u> Let $\varphi_1,\ldots,\varphi_n$ be the elements of some basis for V. We first simplify the basis. Let m be the highest total degree of the φ_i. Letting $\deg_t$ denote the total degree of a polynomial, select those φ_i with $\deg_t \varphi_i=m$. Assume that these are the first r elements $\varphi_1,\ldots,\varphi_r$. Let (i_1,k_1) with $i_1+k_1=m$ be the powers of a monomial $x^{i_1}y^{k_1}$ appearing in φ_1 with non-zero coefficient. By taking linear combinations of φ_1 and φ_j, $j=2,\ldots,r$, we can eliminate this monomial from $\varphi_2,\ldots,\varphi_r$. Give the new basis elements the same names as the old ones. By repeating this process, we arrive at s basis elements $s\leq r$ with the property that each φ_j, $j=1,\ldots,S$ contains a monomial $x^{i_j}\, y^{k_j}$ which does not appear in the others. The number of basis elements of total degree m may decrease since the degree of one of the basis elements may be

reduced by these linear transformations. In this case, we renumber the basis so that the elements of total degree m are at the beginning and continue the process only with these elements.

We will now show that only one monomial of total degree m can appear with non-zero coefficient in each of these basis elements. For suppose this was not true. We apply the transformation $T : x \to \frac{1}{2}x$, $y \to y$ to V. By assumption, TV=V. Therefore there are numbers d_j with

$$\varphi_1(x,y) = \sum_{j=1}^{n} d_j \; T\, \varphi_j(x,y). \qquad (5)$$

We concentrate on the coefficients of the monomials of degree m. Denote by $\hat{\varphi}_j$ that part of φ_j containing the monomials of degree m. Then

$$\varphi_j(x,y) = \sum_{l=o}^{m} a_{jl}\, x^l\, y^{m-l} \qquad j=1,2,\ldots,s.$$

By assumption, $a_{ji_j} \neq 0$ and $a_{ji_t} = 0$ for $t \neq j$. From (5), it follows that

$$\varphi_1(x,y) = \sum_{l=o}^{m} \left[\sum_{j=1}^{n} d_j a_{jl} \left(\tfrac{1}{2}\right)^l \right] x^l\, y^{m-l}$$

$$= \sum_{l=o}^{m} \left[\sum_{j=1}^{s} d_j a_{jl} \left(\tfrac{1}{2}\right)^l \right] x^l\, y^{m-l}.$$

The second equality follows from $T\hat{\varphi}_j = \widehat{T\varphi}_j$. Equating the coefficients of the corresponding powers,

$$a_{1l} = \sum_{j=1}^{s} d_j a_{jl} \left(\tfrac{1}{2}\right)^l, \qquad l=o,\ldots, m.$$

As we noted above, $a_{ji_t} \neq o$ only if $t = j$.

Thus

$$a_{1i_t} = o = \sum_{j=1}^{s} d_j \, a_{ji_t} \, (\tfrac{1}{2})^{i_t}$$

$$= d_t \, a_{i_t i_t} (\tfrac{1}{2})^{i_t} \qquad t=2,\ldots, r.$$

Since the last two factors of the product are non-zero, $d_t = o$ for $t = 2,\ldots,s$ and so the only way to retrieve $\hat{\varphi}_1$ from TV is of the form $C\varphi_1$. But this is manifestly impossible under the assumption that two of the coefficients a_{j1} are non-zero for the transformation T changes their quotient.

We may conclude that, after the simplification, those basis elements which contain a monomial of total degree m contain exactly one and that they are all different. This procedure is repeated in the following way for lower powers. Let $o \le l \le m-1$. We select those φ_j which are of total degree exactly l. With these φ_j, we repeat the above procedure. We end up with, say, $\psi_1,\ldots,\psi_p$ each of which contains exactly one monomial of degree l and none of higher degree and such that the monomials are all different. Using the Ψ_j, we may eliminate the monomials occurring among them from the rest of the φ_j. Then we repeat the procedure again with the Ψ_i, $i=1,\ldots,p$ and $\hat{\varphi}_j$, $j=p+1,\ldots,n$ where now $\hat{\varphi}_j$ is the part of φ_j of total degree exactly l. This guarantees that the degrees of the basis elements is never increased in the simplification procedure.

We obtain a basis of elements such that for each l, $l=o,1,\ldots, m$, each monomial of total degree l can appear at most once in any basis element. For example, $\varphi_1(x,y) = 2x^2y^{2-m} - 3x^{m-4} + 4$.

The next step is to show that, in fact, each of the basis elements must be a monomial. We may write

$$\varphi_j(x,y) = \sum_{l=o}^{m} b_{jl} x^{i_l} y^{l-i_l} . \tag{6}$$

Then, T being the same transformation as before,

$$T\varphi_j \;(x,y) = \sum_{l=o}^{m} b_{jl} \; (\tfrac{1}{2})^{i_l} \; x^{i_l} y^{l-i_l} . \tag{7}$$

By assumption, φ_j is a linear combination of the $T\varphi_j$. But the monomials occuring are mutually disjoint. Thus the only possibility is that $\varphi_j = cT\varphi_j$. But this is only possible if the powers of x appearing in (7) are all the same (or the corresponding coefficient $b_{jl}=o$). In this case, we apply the transformation T: $x \to x$, $y \to \frac{1}{2}y$. Since the powers of y appearing in (7) cannot also be the same, the ratios of the coefficients will have changed unless φ_j is a monomial.

The last step is to show that if $\varphi_j(x,y) = cx^s y^t$, then there are basis vectors which are multiples of $x^i y^k$ for $(i,k) \leq (s,t)$, ie. $V = P_S$ for some lower set S. Let $T : x \to x+h$, $y \to y+h$. Since $TV = V$, $T\varphi_j$ must be a linear combination of the φ_j;

$$T\varphi_j(x,y) = c(x+h)^s (y+k)^t$$

$$= \sum_{l=1}^{n} C_l \; \varphi_l(x,y).$$

But the φ_l are monomials and since all powers $(i,k) \leq (s,t)$ appear in the expansion of $T\varphi_j$, they must also occur among the φ_l. This completes the proof.

The interpolation spaces used in the theory of finite elements are usually translation invariant but not under contractions along the coordinate axes and are therefore not of type P_S for some lower set S. In special cases, for example if one of the sides of a triangular finite element is parallel to one of the coordinate axes, the interpolating space may be a P_S. These remarks also hold for the spaces appearing in Newton-type interpolation methods.

We continue with some notation. Let a lower set S be fixed. Let an incidence matrix E and a point set $Z = \{z_1, \ldots, z_m\}$ be given. By a "row" E_q of E, we mean the two dimensional matrix

$$E_q = (e_{q,i,k})_{(i,k) \in S}$$

of interpolation conditions at the point z_q. We often use the notation $E = E_1 \oplus \ldots \oplus E_m$. By the support A_q of the row E_q, we mean the set

$$A_q = \{(i,k) \mid e_{q,i,k} = 1\}.$$

For $A \subset S$, we define $|E_A|$, the number of interpolation conditions occuring in A, by

$$|E_A| = \sum_{q=1}^{m} \sum_{(i,k)\in A} e_{q,i,k} .$$

A matrix E is called normal if $|E_S| = |S|$ where $|S|$ is the number of points in S. E satisfies the Pólya conditions if $|E_A| \geq |A|$ for each lower set $A \subset S$.

We can now characterize all regular interpolation problems. To do this, we need the definition of an Abel matrix. A matrix E is called an Abel matrix if it is normal and if for each $(i,k) \in S$, the equation $e_{q,i,k} = 1$ is satisfied for exactly one q. Alternatively, $|E_A| = |A|$ for each subset $A \subset S$. A special case is the "Taylor matrix" when m=1 and all $e_{1,i,k} = 1$, $(i,k)\in S$.

Theorem 2 An interpolation matrix E is regular if and only if it is an Abel matrix.

For the proof, see [7].

Whereas in the univarate case the Pólya conditions are necessary and sufficient for conditional regularity, this is not true in two dimensions. We do have

Theorem 3 A normal matrix can be conditionally regular only if it satisfies the Pólya conditions.

Proof Suppose that $|A| = |E_A|$ for some lower set A. We consider a polynomial

$$P(x,y) = \sum_{(i,k)\in A} a_{ik} x^i y^k \tag{8}$$

and the interpolation problem (2) with all $c_{q,i,k} = o$. The homogeneous equations (2) for $(i,k)\in A$ are $|E_A|$ in number while the number of coefficients is $|A| < |E_A|$. Thus there is a nontrivial polynomial (8) which satisfies the conditions. Moreover, if $(s,t) \in S \setminus A$ and $(i,k) \in A$, then either $s>i$ or $t>k$. In the first case, $\partial^s/\partial x^s\ (x^i y^k) = o$. In the second case, $\partial^t/\partial y^t\ (x^i y^k) = o$. Hence $\partial^{s+t}/\partial x^s \partial y^t\ (P) = o$ for $(s,t) \notin A$. It follows that P satisfies the homogeneous conditions (2) for all selections of knots and that E is singular

A simple example shows that the Pólya conditions are not sufficient for the conditional regularity of a normal matrix. Let $E = E_1 \oplus E_2$ where

$$E_1 = E_2 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 1 & 1 & 0 \end{pmatrix}. \tag{9}$$

In this notation, the rows k are numbered from the bottom to the top $k=o,1,\ldots$ and the columns i are numbered from the left to the right $i=o,1,\ \ldots$. We interpolate from P_S where S is the triangle $i+k\leq 2$. Although this seems to be a reasonable problem, it is regular for no choice of points z_1, z_2.

As in the theory of univariate Birkhoff interpolation [8], we are oriented more towards the incidence matrix E and less towards the determinant $D(E,Z)$ of the system of interpolation equations. We also use similar techniques. We manipulate E with shifts of 1's and coalescences of rows. These manipulations correspond to the differentiation of $D(E,Z)$ and to the coalescence of the points of Z.

Let $E = E_1 \oplus E_2 \oplus \ldots \oplus E_m$ be an interpolation matrix. Let $A_q = A(E_q)$ be the support of the row E_q and $B_q = S \setminus A_q$. A shift Λ of E_q takes a one, $e_{q,i,k} = 1$, into position $(i+1,k)$ or $(i,k+1)$ in row q. It is permitted if this position was occupied by a zero. The shift transforms E_q into ΛE_q and E into ΛE. A multiple shift Λ^* of order (α,β) is a repeated application of $\alpha+\beta$ shifts, α of them to the right and β upwards.

Let $z=(x,y)$ be variable and $z_2,\ldots,z_m$ be fixed. Then the derivatives of the determinant $D(E,Z)$ evaluated at $z = z_2$ are given by

$$\left.\frac{\partial^{\alpha+\beta} D}{\partial x^\alpha\ \partial y^\beta}\right|_{z=z_2} = \sum D(\Lambda^*E,\ Z^*)$$

where $Z^* = (z_2, \ldots, z_m)$. The sum is taken over all shifts Λ^* of order (α,β) of E_1 which take its support A_1 into some subset B^* of $B_2 = S \setminus A_2$ and for which $\Lambda^* E$ is a Pólya matrix.

The reason for this is that each derivative of order 1 of $D(E,Z)$ is the sum of determinants obtained by differentiating the rows of the interpolating matrix successively. Each such differentiation corresponds to a shift of a one in A_1 either to the right or upwards. If $A(\Lambda^* E_1)$ and $A(E_2)$ have a common intersection, then the interpolation matrix has two identical rows and so $D(\Lambda^* E, Z^*) = o$. If $\Lambda^* E$ does not satisfy the Pólya conditions, then by Theorem 3 $\Lambda^* E$ is singular and so also $D(\Lambda^* E, Z^*) = o$.

As mentionned before, $D(E,Z)$ is a polynomial in the variables x_i, y_i. We have fixed $z_2, \ldots, z_m$ and set $z_1 = z$. $D(E,Z)$ (z) then has a finite Taylor expansion about $z_2 = (x_2, y_2)$ of the form

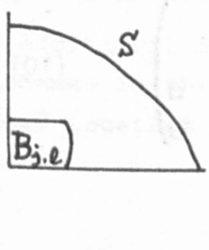
$$D(E, z_2, \ldots, z_m)\ (x,y) = \sum_{i,k} a_{ik}\ (x-x_2)^i (y-y_2)^k$$

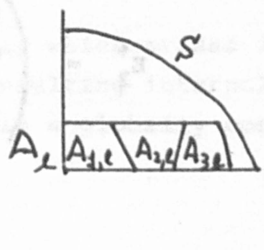
where

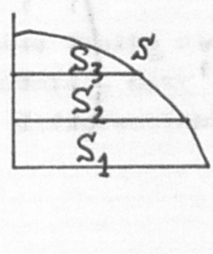
$$i!k!\ a_{ik} = \frac{\partial^{i+k}}{\partial x^i \partial y^k} D(E,Z) \Bigg|_{z=z_2}.$$

If we can show that $a_{ik} \neq o$ for some i,k, then E is conditionally regular. This is easy to verify if the multiple shifts contain only shifts in one direction.

<u>Theorem 4 (Decomposition Theorem)</u> Let E be a matrix $E = \bigoplus \sum E_{j,l}$ with rows $E_{j,l}$, $j,l=1,2,\ldots$. Let $E_{j,l}$ have support $B_{j,l}$. Then E is conditionally regular if S is the disjoint union of S_j where each S_j is an upward (multiple) shift of A_l, $l=1,2,\ldots$ and each A_l is in turn the disjoint union of $A_{j,l}$ with $A_{j,l}$ being a right (multiple) shift of $B_{j,l}$.







The decomposition theorem is proved by shifting the $B_{j,l}$'s to the $A_{j,l}$'s successively by means of shifts to the right. After each multiple shift Λ of order, say (α,o), one verifies that the coefficient $a_{\alpha,o}$ of a Taylor expansion is not zero. Once the strips A_l are filled, they are shifted upward and again one may verify that some coefficient of the relevant Taylor expansion does not vanish. This then guarantees the conditional regularity.

In some cases, points Z can be found for which (E,Z) is regular.

<u>Theorem 5</u> Let $E = E_1 \oplus E_2 \oplus E_3$ with

$$E_1 = E_2 = E_3 = \begin{pmatrix} o & o & & \\ o & o & & \\ 1 & 1 & o & o \\ 1 & 1 & o & o \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The interpolation problem (E,Z), with $Z=\{z_1, z_2, z_3\}$, $z_i = (x_i,y_i)$, is regular for any z_i with $x_1=x_2$, $x_1 \neq x_3$, $y_1 \neq y_2$.

<u>Proof</u> The Taylor expansion of D(E,Z) about z_1 with z_2 as a variable must have the form

$$D(E;z_1,z_3)(x_2,y_2) = \varepsilon_1 D(E^*; z_1,z_3)(y_2-y_1)^8 + (x_2-x_1)P(z_1,z_2,z_3)$$

where E^* is the 2-row matrix whose first row is obtained by shifting A_2 high enough upwards so that it does not intersect A_1. Its second row is E_3.

$$E_1^* = \begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & o & o \\ 1 & 1 & o & o \end{pmatrix} \qquad E_2^* = \begin{pmatrix} o & o & & \\ o & o & & \\ 1 & 1 & o & o \\ 1 & 1 & o & o \end{pmatrix} \qquad (10)$$

The shift reguires eight y-shifts and no x-shifts and corresponds to 8 differentiations with respect to y_2. Any other multiple shift Λ^* involving less than eight y-shifts and no x-shifts leads to a zero contribution to the Taylor expansion as $\Lambda^* E_2$ would still intersect E_1 (have 1's at the same position). Any other multiple shift involving more than eight y-shifts also leads to a zero contribution since then necessarily one of the rows of the interpolation matrix is annihilated. ε_1 is the number of ways the differentiations can be carried out.

If we chose $x_1 = x_2$, $y_1 \neq y_2$, then

$$D(E; z_1, z_3)(x_2, y_2) = \varepsilon_1 D(E^*; z_1, z_3)(y_2 - y_1)^8 .$$

Now we consider the Taylor expansion of $D(E^*; z_1, z_3)$. The only way to shift A_2^* in (10) so that it does not intersect A_1^* and still remains inside of S is to move it to the right far enough. This requires exaclty 8 x-shifts and is the only shift which leads to a non-zero determinant. Thus the Taylor expansion of $D(E^*; z_1, z_3)$ consists of one term. We have

$$D(E^*; z_1, z_3) = \varepsilon_2 \, D(S; z_1)(x_3 - x_1)^8$$

where again ε_2 is the number of different orders in which the differentiations can be carried out. By S, we mean here the one point incidence matrix with entries 1 for all $(i,k) \in S$. Thus S is an Abel matrix and by Theorem 2, $D(S; z_1)$ never vanishes. Since ε_1, ε_2, $x_3 - x_1$ and $y_2 - y_1$ are non-zero, the Theorem is proved.

The regularity of the Bogner-Fox-Schmidt rectangle of finite elements [2] can be proved the same way. Indeed, the regularity of almost all finite element interpolation schemes with three or four interpolation points follow from the above considerations. The exceptions are the cases in which a directional derivative in a non-axial direction is involved.

An additional difficulty which arises in developping finite element schemes is that the resulting interpolating polynomials must be pieced together to obtain a globally continuous or differentiable function. Two examples which yield global continuity are

1) S is the triangle $i+k \leq 3$ without the point (2,1). $Z = \{(x_1, y_1), (x_1,y_2), (x_2,y_3)\}$ with $y_1 \neq y_3$ and $x_1 \neq x_2$. The interpolation conditions are the value, $\partial/\partial x$ and $\partial/\partial y$ at each point. In our notation

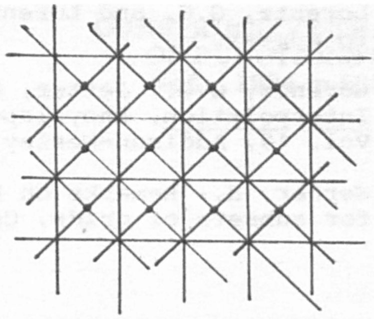
$$E_1 = E_2 = E_3 = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 0 & & \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

2) S is the triangle $i+k \leq 4$ together with the points (5,0) and (0,5). $Z = \{(x_1,y_1), (x_1,y_2), (x_2,y_1)\}$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. These are the vertices of a right triangle. The interpolation conditions are value, $\partial/\partial x$, $\partial/\partial y$, $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ at (x_1,y_1). At the other two points, the value and all derivatives of order up to two are prescribed.

$$E_1 = \begin{pmatrix} 0 & & & & & \\ 0 & & & & & \\ 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad E_2 = E_3 = \begin{pmatrix} 0 & & & & & \\ 0 & & & & & \\ 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ 1 & 1 & 0 & 0 & & \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & & & & & \\ 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Both examples yield globally continuous functions. If the grid points are taken to be as to the right, then the function is even C^1 across vertical and horizontal lines. The grid points which are emphasized correspond to (x_1, y_1).



In concluding, we would like to point out that the examples given here derive from the relatively simple situation in which x-shifts and y-shifts were applied separately. For this reason, there is a resemblance between them and those coming from the Newton-like methods. Examples obtained by using multiple shifts containing both x and y-shifts will be different.

Another point to be made is that it is always clear which space of polynomials we are interpolating from. When using Newton-type interpolation, the spaces must be determined in addition and this could be difficult if many interpolation points are involved.

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