КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 563—568

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.563-568

THE INTERSECTION OF GRONWALL METHODS WITH NÖRLUND AND RIESZ METHODS

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1. Introduction

The definition of a Gronwall method (G,f,g) (see [2]) is based on two functions f and g having the following properties.

1. The function f is holomorphic in $|z| \le 1$ except possibly at $z = 1$ and $w = f(z)$ maps $|z| < 1$ one-to-one onto a domain D interior to $|w| < 1$ in such a manner that $f(o) = 0$, $f(1) = 1$. It is clear that $f'(o) \neq 0$. The inverse function $z = f^{-1}(w)$ is holomorphic on ∂D except possibly at $w = 1$. In a neighborhood of $w = 1$ the inverse function satisfies

$$1-z = (1-w)^{\lambda} \sum_{\nu=0}^{\infty} a_{\nu}(1-w)^{\nu} \qquad (\lambda \ge 1,\ a_0 > 0).$$

2. The function g has the form

$$g(z) = \frac{1}{(1-z)^{\alpha}} + \gamma(z) \qquad (\alpha \ge 0)$$

where $g(z) \neq 0$ in $|z| < 1$ and γ is holomorphic in $|z| \le 1$. The power series $g(z) = \sum_{\nu=0}^{\infty} b_{\nu} z^{\nu}$ satisfies $b_{\nu} \neq 0$ for $\nu = 0,1,\ldots$.

The Gronwall method (G,f,g) is given by the lower triangular matrix $A = (\alpha_{n\nu})$ whose coefficients are defined by the identity

$$\{1-f(z)\}g(z)\{f(z)\}^{\nu} = \sum_{n=\nu}^{\infty} \alpha_{n\nu} b_n z^n \qquad (\nu = 0,1,\ldots).$$

Let $p = \{p_{\nu}\}$ be a fixed sequence of complex numbers and suppose $P_n := \sum_{\nu=0}^{n} p_{\nu} \neq 0$ for all $n = 0,1,\ldots$. The Riesz method (R,p) is defined by the lower triangular matrix $B = (\beta_{n\nu})$ with

$$\beta_{n\nu} = \begin{cases} p_\nu/P_n & \text{if } 0 \le \nu \le n \\ 0 & \text{if } \nu > n \end{cases}.$$

The Nörlund method (N,p) is defined by the lower triangular matrix $C = (\gamma_{n\nu})$ with

$$\gamma_{n\nu} = \begin{cases} p_{n-\nu}/P_n & \text{if } 0 \le \nu \le n \\ 0 & \text{if } \nu > n \end{cases}.$$

If $p_n = \binom{n+r-1}{n}$ where $-r \notin \mathbb{N}$ then the Nörlund method (N,p) gives the Cesàro method (C,r).

It is the object of this paper to establish necessary and sufficient conditions for a summability method to be

a) both (G,f,g) and (N,p) ,

b) both (G,f,g) and (R,p) .

2. The Intersection of Gronwall methods and Nörlund methods

In the following result we give necessary and sufficient conditions for a method A to be a Gronwall method (G,f,g) and also a Nörlund method (N,p).

Theorem 1. The following three statements are equivalent:

(1) A is both (G,f,g) and (N,p).

(2) A = (N,p) where $P(z) := \sum_{n=o}^{\infty} P_n z^n \neq 0$ in $|z| < 1$ and

$$P(z) = \frac{c}{(1-z)^\beta} + \delta(z)$$

where $\beta > 0$, $c \in \mathbb{C}$, $c \neq 0$ and δ is holomorphic in $|z| \le 1$.

(3) A = (G,f,g) where f(z) = z and g satisfies the above requirements.

Proof.

1. Suppose $A = (\alpha_{n\nu})$ is a Gronwall method (G,f,g) and also a Nörlund method (N,p). From

$$F_\nu(f,g;z) := \{1-f(z)\}g(z)\{f(z)\}^\nu = \sum_{n=\nu}^{\infty} \alpha_{n\nu} b_n z^n$$

we obtain with the abbreviation $a := f'(o)$ for $\nu = 0,1,...$

$$\frac{p_o}{P_\nu} = \alpha_{\nu\nu} = a^\nu \frac{b_o}{b_\nu} \neq 0$$

and therefore $b_\nu = \frac{b_o}{p_o} a^\nu P_\nu$ which implies

$$F_\nu(f,g;z) = \sum_{n=\nu}^{\infty} \frac{p_{n-\nu}}{P_n} \frac{b_o}{p_o} a^n P_n z^n = \frac{b_o}{p_o} \sum_{n=\nu}^{\infty} p_{n-\nu} (az)^n .$$

For $\nu = 0$ and $\nu = 1$ we obtain

$$F_1(f,g;z) = \frac{b_o}{p_o} az \sum_{n=o}^{\infty} p_n \cdot (az)^n = az\, F_o(f,g;z) .$$

Therefore we have $f(z) = az$ and hence $f(z) = z$. This gives for $|z| < 1$

$$(2.1) \qquad g(z) = \frac{F_o(f,g;z)}{1-z} = \frac{b_o}{p_o} \sum_{n=o}^{\infty} z^n \cdot \sum_{n=o}^{\infty} p_n z^n = \frac{b_o}{p_o} \sum_{n=o}^{\infty} P_n z^n .$$

So we have (1) ⇒ (2).

2. Let (N,p) be a Nörlund method and suppose that $P(z)$ satisfies the conditions in (2). If we define

$$(2.2) \qquad f(z) := z , \qquad g(z) := \frac{1}{c} \cdot P(z)$$

then it is clear that f,g generate a Gronwall method (G,f,g). We have $b_n = \frac{1}{c} P_n \neq 0$ for all $n = 0,1,\ldots$; if we define $P_{-1} := 0$ we conclude from

$$F_\nu(f,g;z) = (1-z)\frac{1}{c}P(z)z^\nu = z^\nu \cdot \sum_{n=o}^{\infty} \frac{P_n - P_{n-1}}{c} z^n =$$

$$= \sum_{n=\nu}^{\infty} \frac{p_{n-\nu}}{c} z^n = \sum_{n=\nu}^{\infty} \frac{p_{n-\nu}}{P_n} \cdot b_n z^n$$

that (2) ⇒ (3) .

3. Suppose (G,f,g) is a Gronwall method where f reduces to the identity $f(z) = z$. We define $b_{-1} := 0$, $p_n := b_n - b_{n-1}$ for $n = 0,1,\ldots$ and obtain $P_n = \sum_{\nu=o}^{n} p_\nu = b_n \neq 0$. Furthermore we compute

$$F_\nu(f,g;z) = z^\nu (1-z) \cdot \sum_{n=o}^{\infty} b_n z^n = z^\nu \cdot \sum_{n=o}^{\infty} p_n z^n =$$

$$= \sum_{n=\nu}^{\infty} \frac{p_{n-\nu}}{P_n} b_n z^n$$

and hence (3) ⇒ (1), which proves our theorem.

As a corollary of theorem 2 we obtain the following result.

<u>Theorem 2.</u> The following three conditions are equivalent:

(1) A is both (G,f,g) and (C,r).

(2) $A = (C,r)$ where $r > -1$.

(3) $A = (G,f,g)$ where $f(z) = z$ and $g(z) = \frac{1}{(1-z)^{r+1}}$ with $r > -1$.

Proof. The Cesàro method (C,r) is the Nörlund method (N,p) with $p_n = \binom{n-r+1}{n}$. We have $P_n = \binom{n+r}{n}$ and

$$P(z) = \sum_{n=0}^{\infty} P_n z^n = \sum_{n=0}^{\infty} \binom{n+r}{n} z^n = \frac{1}{(1-z)^{r+1}}$$

satisfies the conditions of theorem 1 if and only if $r > -1$.

3. The intersection of Gronwall methods and Riesz methods

In the following result we give necessary and sufficient conditions for a method A to be a Gronwall method and also a Riesz method.

Theorem 2. The following three conditions are equivalent:

(1) A is both (G,f,g) and (R,p).

(2) $A = (R,p)$ where $p_n = q^n$ with $q = 1$ or $|q| > 1$.

(3) $A = (G,f,g)$ where $f(z) = z$ and $g(z) = \frac{1}{(1-z)^2}$ or $g(z) = \frac{1}{1-z} + \frac{1}{z-q}$ with $|q| > 1$.

Proof.

1. Suppose $A = (\alpha_{n\nu})$ is a Gronwall method (G,f,g) and also a Riesz method (R,p). As in the prove of theorem 1 we have $\alpha_{\nu\nu} = a^{\nu}\frac{b_o}{b_{\nu}} \neq 0$ for all ν (where $a := f'(o)$) and since $\alpha_{n\nu} = \frac{p_{\nu}}{P_n}$ this implies $p_n \neq 0$ for $n = 0,1,\ldots$. Without loss of generality we may suppose $p_o = 1$. From the functional equation

$$F_o(f,g;z)\cdot F_{\nu+1}(f,g;z) = F_1(f,g;z)\cdot F_{\nu}(f,g;z)$$

we obtain by comparing the coefficient of z on both sides

$$\frac{\alpha_{10}}{\alpha_{11}} + \frac{\alpha_{\nu+2,\nu+1}}{\alpha_{\nu+2,\nu+2}} = \frac{\alpha_{21}}{\alpha_{22}} + \frac{\alpha_{\nu+1,\nu}}{\alpha_{\nu+1,\nu+1}} .$$

An induction argument shows that for $n \geq 0$

$$\frac{\alpha_{n+1,n}}{\alpha_{n+1,n+1}} = n\frac{\alpha_{21}}{\alpha_{22}} - (n-1)\frac{\alpha_{10}}{\alpha_{11}} .$$

Since $\alpha_{n\nu} = \frac{p_\nu}{p_n}$ this implies for $n \geq 0$

$$(3.1) \qquad \frac{p_n}{p_{n+1}} = n\frac{p_1}{p_2} - (n-1)\frac{1}{p_1} .$$

The solutions of this equations are for $n \geq 0$ of the form

$$(3.2) \qquad p_n = q^n \qquad \text{with } q \neq 0$$

or

$$(3.3) \qquad p_n = \frac{r^n}{\binom{n+s}{n} n!} \qquad \text{with } r \neq 0 \text{ and } -s \notin \mathbb{N}.$$

Using $b_n = \frac{a^n b_o p_n}{p_n}$ we obtain

$$F_o(f,g;z) = \{1-f(z)\}g(z) = b_o \cdot \sum_{n=o}^{\infty} \frac{a^n}{p_n} z^n$$

and this power series must have a radius of convergence $R \geq 1$; but this is impossible if (3.3) holds. So (3.2) holds, we have $p_n = q_n$ and the $\alpha_{n\nu}$ are given by

$$\alpha_{n\nu} = \begin{cases} \frac{1}{n+1} & \text{if } q = 1 \\ \frac{q^\nu (1-q)}{(1-q)^{n+1}} = \frac{1-\frac{1}{q}}{q^{n-\nu}(1-\frac{1}{q^{n+1}})} & \text{if } q \neq 1 \end{cases} .$$

This means that $A = (\alpha_{n\nu})$ is a Nörlund method $(N,\tilde{p})$ generated by the sequence $\tilde{p} = \{\frac{1}{q^n}\}$. Theorem 1 implies $f(z) = z$ and with (2.2) we obtain if $q = 1$

$$P(z) = \sum_{n=o}^{\infty} P_n z^n = \sum_{n=o}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}$$

and $c = 1$. If $q \neq 1$ we get

$$P(z) = \sum_{n=o}^{\infty} \frac{q^{n+1}-1}{q^n(q-1)} z^n$$

and this power series must have radius of convergence $R = 1$ which is impossible if $|q| < 1$. So we obtain in this case $|q| \geq 1$ and for $|z| < 1$ we have

$$P(z) = \frac{q}{q-1} \cdot \frac{1}{1-z} + \frac{q}{q-1} \cdot \frac{1}{z-q} .$$

By theorem 1 this equation implies $|q| > 1$, $c = \frac{q}{q-1}$. Using (2.2) we obtain

$$g(z) = \begin{cases} \frac{1}{(1-z)^2} & \text{if } q = 1 \\ \frac{1}{1-z} + \frac{1}{z-q} & \text{if } |q| > 1 \end{cases} .$$

We therefore proved (1) ⇒ (2) ⇒ (3).

2. Suppose (G,f,g) is the Gronwall method with $f(z) = z$ and $g(z) = \frac{1}{(1-z)^2}$. Then by theorem 2 we obtain $(G,f,g) = (C,1) = (R,p)$ with $p_n = 1$ for all $n \geq 0$.

If (G,f,g) is the Gronwall method with $f(z) = z$ and $g(z) = \frac{1}{1-z} + \frac{1}{z-q}$ with $|q| > 1$ it is easily computed that $(G,f,g) = (R,p)$ with $p_n = q^n$ for all $n \geq 0$.

This completes the prove.

4. References.

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