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KOROVKIN THEOREM IN REARRANGEMENT INVARIANT FUNCTION SPACES

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Summary. A quantitative version of the Korovkin theorem in separable rearrangement invariant function spaces on finite interval is given, using the averaged modulus of continuity.

1. Introduction. Averaged moduli of continuity τ_p in $L_p(0,1)$, $1 \le p < \infty$ (see e.g. [5]) were systematically used last years in the study of approximation problems in which the natural class of consideration is that of all bounded measurable functions on a finite interval. In 1981 Popov [4] proved a quantitative Korovkin theorem in $L_p(a,b)$, $1 \le p < \infty$, in terms of τ_p. Our aim is to extend this result to separable rearrangement function spaces (r.i.f.spaces) on finite interval.

2. Definitions, notations, main result. For the definition of r.i.f. space X on [0,1] we refer to [2] . If $f \in X$, then the right continuous inverse f^* of the distribution function $d_{|f|}(t) = \mu(\{\omega \in [0,1] ; |f(\omega)| > t\})$:

$$f^*(s) = \inf\{t > 0 ;\ d_{|f|}(t) \le s\}, \quad s \in [0,1]$$

is called decreasing rearrangement of f. An order-like relation in $L_1(0,1)$, introduced in [1] , is the following:
for $f, g \in L_1(0,1)$ we write $f \prec g$ if for every $s \in [0,1]$

$$\int_0^s f^*(t)\,dt \le \int_0^s g^*(t)\,dt .$$

The local modulus of continuity

$$\omega_\delta f(t) = \sup\{|f(t')-f(t'')|;\ t', t'' \in [t-\delta/2, t+\delta/2] \cap [0,1]\}, \quad t \in [0,1]$$

can be considered as an operator in the space of all bounded measurable functions on [0,1] . Then we define the averaged modulus of continuity $\tau_X f$ in $(X, \|\cdot\|)$ for the bounded function $f \in X$ in the following way:

$$\tau_X f(\delta) = \|\omega_\delta f\| .$$

Remark 2.1. It is known (see [5]) that $\lim_{\delta\to 0} \tau_1 f(\delta)=0$ iff f is Riemann integrable on [0,1] . On the other hand $\lim_{\delta\to 0} \tau_\infty f(\delta)\neq 0$ if f is not continuous and thus it is natural to consider τ_X only for separable r.i.f. spaces on [0,1] .

Remark 2.2. Instead of [0,1] one can consider an arbitrary finite interval [a,b] . The basic properties of τ_X sre similar to those of τ_p (see e.g. [5] ,p.23-26). We shall use only the inequality $\tau_X f(n\delta) \le n\tau_X f(\delta)$, which can be proved exactly as for L_p .

The main result is the following theorem:

Theorem 2.3. Let L be a positive linear operator in the separable r.i.f. space ($X, \|\cdot\|$) on [0,1] with:

(1) $$Lf_i = f_i + \alpha_i \ , \ f_i(t)=t^i, \ i=0,1,2 \ ; \ \alpha_0(t)\equiv 0$$

and $d=\max\{|\alpha_2(t)-2t\alpha_1(t)| \ ; \ t\in[0,1]\}$.

Then for every bounded function $f\in X$

$$\|Lf-f\| \le c\tau_X f(\sqrt{d}),$$

where c is absolute constant.

3. Auxiliary results. We need some lemmas for the proof of Theorem 2.3. In the sequell ($X, \|\cdot\|$) is a separable r.i.f. space on [0,1] .

Lemma 3.1. ([5] , p.111). If L is a positive linear operator in X which satisfies (1) then for every $c\in(0,1)$

$$|L\chi_{[c,1]}(t)-\chi_{[c,1]}(t)| \le d/(t-c)^2, \ t\in[0,1], \ t\neq c,$$

where as usual $\chi_{[c,1]}$ is the characteristic function of the interval [c,1] .

For a natural n denote $x_i=i/n$, $i=0,1,\dots,n$;

$$\Delta(i)=[x_{i-1},x_i), \ i=1,2,\dots,n-1, \ \Delta(n)=[x_{n-1},x_n] .$$

Lemma 3.2. ([3]). The averaging operator

$$T_n f = n\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(t)\,dt\,\chi_{\Delta(i)}$$

in X is bounded and $\|T_n\|=1$ for every n .

Denote by D_n the set of all linear operators $T=\{\alpha_{ij}\}_{i,j=1}^{n}$ in R^n such that the $n\times n$ matrix $\{\alpha_{ij}\}$ has the properties: $\sum_{i=1}^{n}|\alpha_{ij}|\le 1$ for every $j\le n$,and $\sum_{j=1}^{n}|\alpha_{ij}|\le 1$ for every $i\le n$.

Lemma 3.3. Let $a=(a_1,a_2,\dots,a_n)$, $b=(b_1,b_2,\dots,b_n)$ are two vectors in R^n with $a=Tb$, $T\in D_n$. Then for the functions $f=\sum_{i=1}^{n}a_i\chi_{\Delta(i)}$, $g=\sum_{i=1}^{n}b_i\chi_{\Delta(i)}$ in X the inequality holds: $\|f\| \leq \|g\|$.

Proof. It is known (see e.g. [3] ,Proposition 2.a.8.) that if $f \prec g$ then $\|f\| \leq \|g\|$ and what we have to prove is that $f \prec g$. From a result of Hardy, Littlewood and Polya (Proposition 2.a.5 in [3]) $T \in D_n$ implies

(2) $\sum_{i=1}^{k} a_i^* \leq \sum_{i=1}^{k} b_i^*$ for every $k \leq n$,

where $(a_1^*, a_2^*, \ldots, a_n^*)$, respectively $(b_1^*, b_2^*, \ldots, b_n^*)$,are the decreasing rearrangement of ($|a_1|, |a_2|, \ldots, |a_n|$) and ($|b_1|, |b_2|, \ldots, |b_n|$).

Obviously $f^* = \sum_{i=1}^{n} a_i^* \chi_{\Delta(i)}$,and $g^* = \sum_{i=1}^{n} b_i^* \chi_{\Delta(i)}$. Now it is easy to check that

(3) $\int_0^s f^*(t)\,dt \leq \int_0^s g^*(t)\,dt$ for every $s \in (0,1]$

i.e. $f \prec g$.

Indeed,let $s \in \Delta(j)$. If $a_j^* \leq b_j^*$,using (2) with $k=j-1$, we obtain:

$$\int_0^s f^*(t)\,dt = \frac{1}{n}\sum_{i=1}^{j-1} a_i^* + a_j^*\left(s-\frac{j-1}{n}\right) \leq \frac{1}{n}\sum_{i=1}^{j-1} b_i^* + b_j^*\left(s-\frac{j-1}{n}\right) = \int_0^s g^*(t)\,dt .$$

To obtain (3) in the case $a_j^* > b_j^*$ we use (2) with $k=j$:

$$\int_0^s f^*(t)\,dt = \frac{1}{n}\sum_{i=1}^{j} a_i^* - a_j^*(s/n-s) \leq -\sum_{i=1}^{j} b_i^* - b_j^*(j/n-s) = \int_0^s g^*(t)\,dt .$$

Thus lemma is proved.

4. Proof of Theorem 2.3. We suppose first that $0 < d \leq 1$ (the case $d>1$ is not interesting) and denote by n the natural number with the property $h = 1/n \leq d^{1/2} \leq 1/(n-1)$.

Following the method from [4] let us consider the functions

$$F = \sum_{i=1}^{n} F_i \chi_{\Delta(i)} ,\quad F_i = \sup\{f(t);\, t \in \Delta(i)\} ,\ i=1,2,\ldots,n ;$$

$$G = \sum_{i=1}^{n} G_i \chi_{\Delta(i)} ,\quad G_i = \inf\{f(t);\, t \in \Delta(i)\} ,\ i=1,2,\ldots,n .$$

Obviously $G \leq f \leq F$,and $|F(t)-G(t)| \leq \omega_{2h} f(t)$. Hence $\|F-G\| \leq \|\omega_{2h} f\| \leq \|\omega_{2\sqrt{d}} f\| = \tau_X f(2\sqrt{d}) \leq 2\tau_X f(\sqrt{d})$.

It is easy to verify that .

$$\|Lf-f\| \leq \|LG-G\| + 2\|F-G\| + 2\|LF-F\| .$$

The last two inequalities show that in order to prove the theorem we have to estimate $\|LF-F\|$ and $\|LG-G\|$.

Let us estimate $\| LF-F \|$. We shall use the following representation for F: $F=\sum_{i=1}^{n}(F_i-F_{i-1})\chi_{[x_{i-1},1]}$, $F_0=0$.

Obviously

(4) $$|LF(t)-F(t)| \le \sum_{i=1}^{n} |F_i-F_{i-1}| \, |L\chi_{[x_{i-1},1]}(t)-\chi_{[x_{i-1},1]}(t)|, \ t\in[0,1].$$

First we observe that

$$|F_i-F_{i-1}| \le \omega_{2h} f(x_{i-1}) = n\int_{x_{i-1}}^{x_i} \omega_{2h} f(x_{i-1})\,dt \le n\int_{x_{i-1}}^{x_i} \omega_{4h} f(t)\,dt$$

for every $i=1,2,\dots,n$.

On the other hand for $t\in \Delta(j)$, according to Lemma 3.1 we have:

$$|L\chi_{[x_{i-1},1]}(t)-\chi_{[x_{i-1},1]}(t)| \le \frac{d}{(t-x_{i-1})^2} \le \frac{4}{(j-i)^2} \quad for \ i<j,$$

$$|L\chi_{[x_{i-1},1]}(t)-\chi_{[x_{i-1},1]}(t)| \le \frac{d}{(t-x_{i-1})^2} \le \frac{4}{(i-j-1)^2} \quad for \ i>j+1.$$

Since for

$$i=j,\ j+1: |L\chi_{[x_{i-1},1]}(t)-\chi_{[x_{i-1},1]}(t)| \le 1$$

from (4) it follows

(5) $$|LF-F| \le \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\alpha_{ij}\,b_i\Big)\chi_{\Delta(j)},$$

where $b_i = n\int_{x_{i-1}}^{x_i}\omega_{4h} f(t)\,dt$, $i=1,2,\dots,n$ and

$$\alpha_{ij}=\begin{cases} 4/(j-i)^2, & i<j \\ 1, & i=j,j+1 \\ 4/(i-j-1)^2, & i>j+1 \end{cases}, \quad i,j=1,2,\dots,n.$$

Obviously $\sum_{i=1}^{n}\alpha_{ij} \le 2\big(4\sum_{k=1}^{\infty}\frac{1}{k^2}+1\big)=c_1$ for every $j\le n$ and $\sum_{j=1}^{n}\alpha_{ij}\le c_1$, for every $i\le n$.

Denote $\beta_{ij}=\alpha_{ij}/c_1$, $i,j=1,2,\dots,n$. Then (5) can be written as:

(6) $$|LF-F| \le c_1\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\beta_{ij}\,b_i\Big)\chi_{\Delta(j)}$$

where the linear operator T associated to the matrix $\{\beta_{ij}\}_{i,j=1}^{n}$ is from D_n. Now from Lemma 3.3. it follows

(7) $$\Big\|\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\beta_{ij}\,b_i\Big)\chi_{\Delta(j)}\Big\| \le \Big\|\sum_{j=1}^{n} b_j\,\chi_{\Delta(j)}\Big\|.$$

The inequalities (6) and (7) imply:

$$\|LF-F\| \le c_1 \Big\| n\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\omega_{4h} f(t)\,dt\,\chi_{\Delta(j)}\Big\|$$

and from Lemma 3.2. it follows

$$\| LF - F \| \leq c_1 \| \omega_{4h} f \| \leq c_1 \| \omega_{4\sqrt{d}} f \| = c_1 \tau_X f(4\sqrt{d}) .$$

Using the inequality from Remark 2.2 we get the desired estimate for $\| LF - F \|$:

$$\| LF - F \| \leq 4 c_1 \tau_X f(\sqrt{d}) .$$

In the same way $\| LG - G \| \leq 4 c_1 \tau_X f(\sqrt{d})$ and finally

$$\| Lf - f \| \leq (4 + 12 c_1) \tau_X f(\sqrt{d}) .$$

Thus Theorem 2.3 is proved with $c = 4 + 12 c_1$.

Remark 4.1. It is easy to verify that $c \leq 4 + 12.46/3 = 188$, instead of 748 obtained in [5] for the case L_p .

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