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POWERS OF BERNSTEIN-TYPE OPERATORS

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1. We will make use of the following equations involving the Stirling numbers of the second kind $\mathfrak{S}_{n,m}$ (see e.g.[9],[10]):

$$(x+1)^n = \sum_{m=0}^{n} \mathfrak{S}_{n+1,m+1} \binom{x}{m},$$

$$\sum_{k=0}^{n} \binom{n+1}{k} \mathfrak{S}_{k,m} = (m+1)\, \mathfrak{S}_{n+1,m+1}$$ (for proofs see [7]).

2. Let V denote a linear operator in a finite dimensional space. The eigenvalues λ_k $(k=\overline{0,m})$, each one of order r_k, of the operator V make the spectrum $\mathfrak{S}(V)$.

If φ is an analytic function upon an open set containing the spectrum $\mathfrak{S}(V)$ then the following Lagrange-Sylvester's formula holds

$$\varphi(V) = \sum_{k=0}^{m} \sum_{j=0}^{r_k-1} \frac{\varphi^{(j)}(\lambda_k)}{j!} H_{k,j}(V),$$

where $H_{k,j}(V) := (V-\lambda_k I)^j H_k(V)$ and $H_k(\cdot)$ is the Hermite interpolating polynomial satisfying the conditions

$$H_k(\lambda_i) = \delta_{i,k} \quad \text{for } i,k=\overline{0,m},$$

$$H_k^{(j)}(\lambda_k) = 0 \quad \text{for } k=\overline{0,m} \text{ and } j=\overline{0,r_k-1}$$ [4].

3. As usual, $p_{m,k}(x) := \binom{m}{k} x^k (1-x)^{m-k}$ with $x \in \langle 0,1 \rangle$ and

$$B_m(f) := B_m f,\quad B_m f(x) := \sum_{k=0}^{m} f\left(\frac{k}{m}\right) p_{m,k}(x).$$

P.C.Sikkema [11] proved that for any natural m the powers $B_m{}^p$ of the Bernstein operator B_m (see e.g.[5]) tend uniformly to B_1 as $p \to \infty$.

In 1930 L.V.Kantorovič defined (see e.g.[5]) the operator K_m which maps any Lebesgue-integrable function $f \in L_{(0,1)}$ into the polynomial $K_m f$ due to the formula

$$K_m f(x) := (m+1) \sum_{k=0}^{m} f_{m,k} p_{m,k}(x)$$

where $f_{m,k} := \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s)\,ds$.

Lately many authors have dealt with the Kantorovič operators K_m (e.g.[1],[2],[6],[8],[12]) that are close to the B_m for $K_m F = (B_{m+1} f)'$ where F denotes an indefinite integral of f, i.e. $F' = f$. Now, from the Sikkema's theorem or from Korovkin's simplification theorem one can easily obtain

Theorem 1. $K_m{}^p \longrightarrow K_0$ as $p \longrightarrow \infty$.

In the next part of this paper we'll give another proof of this result, in Part 5 we'll extend the presented method to some Bernstein-Stancu operators (for the excellent survey of them see [3]).

4. Fix a natural m and write K instead of K_m.

At any point $x \in \langle 0,1 \rangle$ we have $Kf(x) = (m+1) A_1{}^T Y$ where

$A_1{}^T$ is the transposition of a column $A_1 = (A_1(0), A_1(1), \ldots, A_1(m))$,

$A_1 := N^{-1} C$,

$N := [\binom{j}{k}]$ is the low triangular matrix of the Newtonian coefficients $\binom{j}{k}$, $j,k = \overline{0,m}$,

$C_\mu := f_{m,\mu}$ is the μ-th component of a vector C, $\mu = \overline{0,m}$,

$Y(k) := \binom{m}{k} x^k$ is the k-th component of a vector Y, $k = \overline{0,m}$.

Because $A_1(k) = \sum_{\mu=0}^{k} (-1)^{k-\mu} \binom{k}{\mu} C_\mu$, so

$$K^2f(x) := K\{Kf(x)\} = K\{(m+1)\sum_{\mu=0}^{m} C_\mu p_{m,k}(x)\} =$$

$$= (m+1)\sum_{\mu=0}^{m} D_\mu p_{m,k}(x) = m+1\, A_2^T Y, \text{ where}$$

$$D_\mu := \int_{\frac{\mu}{m+1}}^{\frac{\mu+1}{m+1}} Kf(s)ds = \sum_{j=0}^{m} A_1(j)\binom{m}{j}(m+1)^{-j-1}\sum_{i=0}^{1}\binom{j+1}{i}\mu^i ,$$

$$A_2(k) := \sum_{\mu=0}^{k}(-1)^{k-\mu}\binom{k}{\mu}D_\mu = \sum_{j=0}^{m} V(k,j)A_1(j),$$

$V(k,j) := B(k,j)V(0,j)$,

$B(k,j) := (k+1)!\,\sigma_{j+1,k+1}$ by Part 1,

$V(0,j) := \lambda_j/(j+1)!$,

$\lambda_j := V(j,j) = (m+1)^{-j}\, m!/(m-j)!$ is the j-th eigenvalue of the upper triangle matrix $V := [V(k,j)],\ k,j=\overline{0,m}$.

This way we obtain the formula

$$K^p f(x) = (m+1)A_p^T Y$$

where $A_p := V^{p-1}A_1$. Hence $\lim_{p\to\infty} K^p f(x) = (m+1)U^T Y$, where

$U := ZA_1 = FC$,

$F := ZN^{-1}$ and

$Z := \lim_{p\to\infty} V^p$,

so our task is resolved into the determining the matrix Z.

The spectrum $\sigma(V)$ has exactly m+1 different positive elements $\lambda_0, \lambda_1, \dots, \lambda_m$ and the greatest one of them is $\lambda_0 = 1 > \lambda_j$ for $j=\overline{1,m}$, so applying Part 2 with $\varphi(\lambda) := \lambda^p$ we get

$$Z = \prod_{j=1}^{m} \frac{V-\lambda_j I}{1-\lambda_j},$$

for the Hermite interpolation is reduced to the Lagrange interpolation.

V is an upper triangular matrix, so $Z(k,j)=0$ for $j=\overline{0,m}$ and $k=\overline{1,m}$. Obviously, $Z(0,0)=1$. Assuming $Z(0,i) = m!/\{(m-i)!\,(i+1)!\}$ for $i=\overline{0,j-1}$, by induction and using formulas from Part 1 we conclude that $Z(0,j) = \binom{m+1}{j+1}/(m+1)$ for $j=\overline{0,m}$. Now it is easy to see that $U = \left({}_0\int^1 f(s)ds / m+1, 0, \ldots, 0 \right)$ what makes the proof complete.

5. The presented method can be applied to the Bernstein operator B_m (its two greatest eigenvalues are equal to 1) as well as to some other Bernstein-type operators, e.g. two Bernstein-Stancu operators L_m and R_m defined as follows

$$L_m f(x) := \sum_{k=0}^{m} f\left(\frac{k}{m+1}\right) p_{m,k}(x),$$

$$R_m f(x) := \sum_{k=0}^{m} f\left(\frac{k+1}{m+1}\right) p_{m,k}(x)$$

provided f has its values at every point given above. Both L_m and R_m can be obtained by applying the trapezoid rule well-known in the numerical integration. Proceeding as in Part 4 we get the following

Theorem 2. $L_m^p f(x) \xrightarrow[p\to\infty]{} f(0)$ and $R_m^p f(x) \xrightarrow[p\to\infty]{} f(1)$ at every point $x \in \langle 0,1 \rangle$.

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