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MEAN CESARO SUMMABILITY OF ORTHOGONAL POLYNOMIALS

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Let $d\alpha$ be a positive measure on the real line with finite moments and infinite support, and let $\{p_n(d\alpha)\}_{n=0}^{\infty}$, $p_n(d\alpha,x) = \gamma_n(d\alpha)x^n + \dots$, $\gamma_n(d\alpha) > 0$, be the system of orthonormal polynomials associated with $d\alpha$. In [1], [3]-[8], [10]-[12] a number of important results were found concerning properties of orthogonal polynomials corresponding to such measures $d\alpha$ for which $\mathrm{supp}(d\alpha) = [-1,1]$ and $\alpha' > 0$ almost everywhere in $[-1,1]$. The purpose of this paper is to show that some of these results remain valid if the condition $\mathrm{supp}(d\alpha) = [-1,1]$ is replaced by the assumption that for every $\varepsilon > 1$ $\mathrm{supp}(d\alpha)\setminus[-\varepsilon,\varepsilon]$ is a finite set. According to Blumenthal's theorem (see e.g. [10, Theorem 3.3.7, p. 23]) if the orthogonal polynomials are generated by the recurrence formula

(1) $$x\,p_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}\ , \qquad n = 0, 1,\dots ,$$

$a_0 = 0$, $a_n > 0$, $n = 1, 2,\dots$, $b_n \in \mathbb{R}$, $p_{-1} = 0$, $p_0 = \gamma_0$, then the support of the corresponding measure has the above described property provided that

(2) $$\lim_{n\to\infty} a_n = 1/2 \quad \underline{\text{and}} \quad \lim_{n\to\infty} b_n = 0\ .$$

For given $d\alpha$ let the Christoffel function $\lambda_n(d\alpha)$, $n = 1, 2,\dots$, be defined by

$$\lambda_n^{-1}(d\alpha) = \sum_{k=0}^{n-1} |p_k(d\alpha)|^2\ .$$

Another equivalent definition can be given by

(3) $$\lambda_n(d\alpha,x) = \min|R(x)|^{-2} \int |R|^2 \, d\alpha$$

where the minimum is taken over all polynomials R of degree less than n [2, Theorem 1.4.1, p. 25]. It was proved in [7, Theorem 12.1] that if $\operatorname{supp}(d\alpha) = [-1,1]$ and $\alpha' > 0$ almost everywhere in $[-1,1]$ then $\alpha' n^{-1} \lambda^{-1}(d\alpha)$ converges to $\pi^{-1}(1-x^2)^{-1/2}$ in $L^1[-1,1]$ when $n \to \infty$. Our main result is the following generalization of this theorem.

THEOREM 1. Suppose that $d\alpha$ is such that α' is positive almost everywhere in $[-1,1]$ and for every $\varepsilon > 1$ $\operatorname{supp}(d\alpha) \setminus [-\varepsilon,\varepsilon]$ is a finite set. Then

(4) $$\lim_{n\to\infty} \int_{-1}^{1} \left| \frac{\alpha'(x)}{n\lambda_n(d\alpha,x)} - \frac{1}{\pi\sqrt{1-x^2}} \right| dx = 0$$

holds.

Proof. Let $\Delta = [-1,1]$ and let v be defined by $v(x) = \pi^{-1}(1-x^2)^{-1/2}$ for $-1 \le x \le 1$. Then by Schwarz's inequality

$$\left(\int_\Delta \left| \frac{\alpha'}{n\lambda_n(d\alpha)} - v \right| \right)^2 = \left(\int_\Delta \left| \sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} - \sqrt{v} \right| \left| \sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} + \sqrt{v} \right| \right)^2 \le$$

$$\le \int_\Delta \left(\sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} - \sqrt{v} \right)^2 \int_\Delta \left(\sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} + \sqrt{v} \right)^2 \le$$

$$\le 2 \int_\Delta \left(\sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} - \sqrt{v} \right)^2 \left(\int_\Delta \frac{\alpha'}{n\lambda_n(d\alpha)} + \int_\Delta v \right) \le$$

$$\le 2 \int_\Delta \left(\sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} - \sqrt{v} \right)^2 \left(\int_{\mathbb{R}} \frac{d\alpha}{n\lambda_n(d\alpha)} + \int_\Delta v \right) =$$

$$= 4 \int_\Delta \left(\sqrt{\frac{\alpha'}{n\lambda_n(d\alpha)}} - \sqrt{v} \right)^2 = 4 \left(\int_\Delta \frac{\alpha'}{n\lambda_n(d\alpha)} - 2 \int_\Delta \sqrt{\frac{\alpha' v}{n\lambda_n(d\alpha)}} + \int_\Delta v \right) \le$$

$$\le 4 \int_{\mathbb{R}} \left(\frac{d\alpha}{n\lambda_n(d\alpha)} - 2 \int_\Delta \sqrt{\frac{\alpha' v}{n\lambda_n(d\alpha)}} + \int_\Delta v \right) = 8 \left(1 - \int_\Delta \sqrt{\frac{\alpha' v}{n\lambda_n(d\alpha)}} \right)$$

so that (4) will be proved if we show the validity of

$$(5) \qquad 1 \le \liminf_{n\to\infty} \int_\Delta \sqrt{\frac{\alpha' v}{n\lambda_n(d\alpha)}} = \ell \ .$$

In order to prove (5) we pick an arbitrary nonnegative continuous function g in Δ and apply Holder's inequality to obtain

$$\left(\int_\Delta (vg\alpha')^{1/4}\right)^4 \le \left(\int_\Delta \sqrt{\frac{\alpha' v}{n\lambda_n(d\alpha)}}\right)^2 \cdot \int_\Delta n\lambda_n(d\alpha) g v^{-1} \ .$$

The function $f = gv^{-1}$ satisfies the conditions of the Lemma which is formulated and proved at the end of this paper. Thus by (23) we have

$$(6) \qquad \left(\int_\Delta (vg\alpha')^{1/4}\right)^4 \le \ell^2 \int_\Delta g v^{-2} d\alpha \ .$$

By Lemma 1 in [5] there exists a sequence $\{h_m\}_{m=1}^\infty$ of continuous functions in Δ such that $0 < h_m(x) \le 1$ for $x \in \Delta$, $\lim_{m\to\infty} h_m(x) = 1$ for almost every x in Δ and $\lim_{m\to\infty} \int_\Delta h_m d\alpha_s = 0$ where $d\alpha_s$ is the singular (nonabsolutely continuous) component of $d\alpha$. Moreover, for fixed $\delta > 0$ we can choose a sequence $\{H_k\}_{k=1}^\infty$ of continous functions in Δ such that $0 < H_k(x) \le 1/\delta$ for $x \in \Delta$ and $\lim_{k\to\infty} H_k(x) = \left(\delta + v^{-3}(x)\alpha'(x)\right)^{-1}$ holds almost everywhere in Δ. For a fixed $\delta > 0$ we can apply (6) with $g = h_m H_k$ and then we can let first $m \to \infty$, then $k \to \infty$, and finally $\delta \to 0$. We obtain

$$\left(\int_\Delta v\right)^4 \le \ell^2 \int_\Delta v \ ,$$

and, since $\int_\Delta v = 1$, inequality (5) follows and so does Theorem 1.

Remark 1. It was proved in [1], [4] and [10] that if the orthogonal polynomials satisfy the recurrence formula (1), and (2) and

$$\sum_{n=1}^{\infty} \left(|a_n - a_{n-1}| + |b_n - b_{n-1}|\right) < \infty$$

hold, then $d\alpha$ does satisfy the conditions of Theorem 1, and, consequently, both Theorem 1 and Theorem 2 below are valid in this case.

In what follows we denote the zeros of $p_n(d\alpha)$ by $x_{kn}(d\alpha)$, $k = 1,2,\ldots,n$. All these zeros are real and simple, and they belong to the smallest interval containing the support of $d\alpha$ [2, Theorem 1.2.2, p. 17]. When using $x + \sqrt{x^2 - 1}$ we mean that branch of this function which is analytic in the complex plane cut along $[-1,1]$ and which is positive for $x > 1$.

THEOREM 2. Let $d\alpha$ satisfy the conditions of Theorem 1. Then

$$\lim_{n\to\infty} \int \frac{1}{n\lambda_n(d\alpha)}\, d\alpha_s = 0 \tag{7}$$

where $d\alpha_s$ is the singular (nonabsolutely continuous) component of $d\alpha$,

$$\lim_{n\to\infty} \int \frac{f}{n\lambda_n(d\alpha)}\, d\alpha = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \tag{8}$$

holds for every function $f \in L^{\infty}_{d\alpha}$,

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{f(x)\alpha'(x)}{n\lambda_n(d\alpha,x)}\, dx = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \tag{9}$$

holds for every function $f \in L^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \tag{10}$$

holds for every bounded function f which is Riemann integrable in $[-1,1]$. Moreover,

$$\lim_{n\to\infty} \gamma_n(d\alpha)^{1/n} = 2, \tag{11}$$

(12) $$\lim_{n\to\infty} \max_{-1 \le x \le 1} |p_n(d\alpha,x)|^{1/n} = 1 ,$$

(13) $$\lim_{n\to\infty} |p_n(d\alpha,x)|^{1/n} = |x + \sqrt{x^2 - 1}|$$

holds uniformly on every compact set K in the complex plane such that $K \cap \operatorname{supp}(d\alpha) = \emptyset$, and

(14) $$\lim_{n\to\infty} |p_n(d\alpha,x)|^{1/n} = |x + \sqrt{x^2 - 1}|^{-1}$$

uniformly on compact subsets of $\operatorname{supp}(d\alpha) \setminus [-1,1]$. If G is a simply connected region in the complex plane such that $G \cap \operatorname{supp}(d\alpha) = \emptyset$ then

(15) $$\lim_{n\to\infty} p_n(d\alpha,x)^{1/n} = x + \sqrt{x^2 - 1}$$

holds uniformly on compact subsets of G . Here we take that branch of the n-th root of $p_n(d\alpha)$ which is positive for large positive values of x .

<u>Proof</u>. Since

$$\int_{\mathbb{R}} \frac{1}{n\lambda_n(d\alpha)}\, d\alpha = 1 ,$$

(7), (8) and (9) are immediate consequences of (4). By Lemma 5.1 in [10, p. 49] if $\operatorname{supp}(d\alpha)$ is compact then

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \int_{\mathbb{R}} \frac{f}{n\lambda_n(d\alpha)}\, d\alpha \right] = 0$$

for every continuous function f . Hence (10) follows from (8) if f is continuous, and then standard one-sided approximation arguments yield (10) in the general case as well. In order to prove (11) and (12) we introduce two measures $d\beta$ and $d\mu$ with supports $[-1,1]$ as follows. The measure $d\beta$ is the Chebyshev measure, that is $d\beta(x) = (1-x^2)^{-1/2}dx$ for $-1 \le x \le 1$, whereas $d\mu(x) = \alpha'(x)dx$, $-1 \le x \le 1$, is the absolutely continuous component of $d\alpha$. Then $\gamma_n(d\alpha) \le \gamma_n(d\mu)$ [14, p. 28] and since

(16) $$\lim_{n\to\infty} \gamma_n(d\mu)^{1/n} = 2$$

[2, p. 128], the inequality

(17) $$\limsup_{n\to\infty} \gamma_n(d\alpha)^{1/n} \le 2$$

follows. Now let $\varepsilon > 1$ be fixed and let $R(x) = \prod_{k=1}^{m} (x - t_k)$ where $\bigcup_{k=1}^{m} \{t_k\} = \operatorname{supp}(d\alpha) \setminus [-\varepsilon,\varepsilon]$. Then by orthogonality

$$\frac{\gamma_{n-m}(d\beta)}{\gamma_n(d\alpha)} = \int_{-\varepsilon}^{\varepsilon} p_{n-m}(d\beta)R\, p_n(d\alpha)d\alpha \le$$

$$\le \max_{-\varepsilon \le x \le \varepsilon} |R(x)| \max_{-\varepsilon \le x \le \varepsilon} |p_{n-m}(d\beta)| (\int_{\mathbb{R}} d\alpha)^{1/2}$$

for $m > m$. However, $\gamma_n(d\beta) = 2^{n-1/2}\sqrt{\pi}$ and

$$p_n(d\beta,x) = \frac{1}{\sqrt{2\pi}} \left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right]$$

for $n = 1, 2, \ldots$ so that

$$\frac{2^n}{\gamma_n(d\alpha)} \le 2^{m+1} \max_{-\varepsilon \le x \le \varepsilon} |R(x)| \cdot (\varepsilon + \sqrt{\varepsilon^2 - 1})^{n-m} (\int_{\mathbb{R}} d\alpha)^{1/2}$$

holds. Taking n-th roots, letting first $n \to \infty$ and then $\varepsilon \to 1$ we obtain

$$\liminf_{n\to\infty} \gamma_n(d\alpha)^{1/n} \ge 2 ,$$

and thus (11) holds by (17). To prove (12) we use (3) to obtain

$$p_n^2(d\alpha) \le \lambda_{n+1}^{-1}(d\alpha) \le \lambda_{n+1}^{-1}(d\mu) ,$$

x real, so that by Christoffel-Darboux's formula [14, p. 43] and Markov's inequality [9, p. 137]

$$\max_{-1 \le x \le 1} p_n^2(d\alpha,x) \le 2(n+1)^2 \max_{-1 \le x \le 1} |p_n(d\mu,x)| \max_{-1 \le x \le 1} |p_{n+1}(d\mu,x)| .$$

Since

$$\limsup_{n\to\infty} \max_{-1 \le x \le 1} |p_n(d\mu,x)|^{1/n} \le 1$$

[2, p. 124] the inequality

$$\limsup_{n\to\infty} \max_{-1 \le x \le 1} |p_n(d\alpha,x)|^{1/n} \le 1 \tag{18}$$

follows. On the other hand,

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\mu)} = \int_{-1}^{1} p_n(d\alpha)p_n(d\mu)d\mu \le \max_{-1 \le x \le 1} |p_n(d\alpha,x)|\left(\int_{-1}^{1} d\mu\right)^{1/2} ,$$

and thus by (11) and (16)

$$\limsup_{n\to\infty} \max_{-1 \le x \le 1} |p_n(d\alpha,x)|^{1/n} \ge 1$$

which combined with (18) proves (12). Now we proceed with proving (13). First we will show that if $x \notin \mathrm{supp}(d\alpha)$ then x is not a point of accumulation of the set of zeros of $p_n(d\alpha)$, $n = 1, 2, \ldots$. Let ω be the smallest closed interval containing $\mathrm{supp}(d\alpha)$. Since all the zeros of each $p_n(d\alpha)$ belong to ω [2, Theorem 1.2.2, p. 17], we may assume $x \in \omega \setminus \mathrm{supp}(d\alpha)$. Then either $x < -1$ or $x > 1$ and x lies in between two isolated mass points of $d\alpha$. Suppose without loss of generality that $x > 1$. Let $t_1 < t_2 < \ldots < t_m$ denote those points of $\mathrm{supp}(d\alpha)$ which belong to (x,∞) . Let $\delta > 0$ be such that

$$x + \delta < t_1 - \delta < t_2 + \delta < \ldots < t_{m-1} + \delta < t_m - \delta < t_m$$

and $[x-\delta,x] \cap \mathrm{supp}(d\alpha) = \emptyset$. By Theorem 6.1.1 in [14, p. 111] there exists $N = N(\delta,\{t_i\})$ such that for every $n \ge N$, $p_n(d\alpha)$ has zeros in each interval $[t_i - \delta, t_i + \delta]$, $i = 1, 2, \ldots, m$. Thus for $n \ge N$ $p_n(d\alpha)$ has not less than m zeros in $[t_i - \delta,\infty)$. On the other hand α takes exactly $m + 1$ values in $(x-\delta,\infty)$ not counting the values $\alpha(t_i)$. Thus by Theorem 1.2.4 [2, p, 18] for every $n \geq 1$, $p_n(d\alpha)$ has no more than m zeros in $(x-\delta,\infty)$. Hence for $n > N$ both $(x-\delta,\infty)$ and $(t_1-\delta,\infty)$ contain exactly m zeros of $p_n(d\alpha)$, that is $[x-\delta,x+\delta]$ contains no zeros of $p_n(d\alpha)$ if $n \geq N$. Hence we have proved our claim concerning $x \notin \mathrm{supp}(d\alpha)$ not being a point of accumulation of the zeros. If K is a compact set in the complex plane and $K \cap \mathrm{supp}(d\alpha) = \emptyset$ then there exists $N = N(K)$ such that for each $x \in K$, $f_1(t) = \log|x-t|$ is a continuous function for $t \in \overline{\bigcup_{n \geq N} \bigcup_{1 \leq k \leq n} \{x_{kn}(d\alpha)\}}$. Hence applying (10) with $f = f_1$ and using (11) the limit relation (13) follows. By (11) for $n \geq N$ the sequence $n^{-1}\log|p_n(d\alpha)|$ is uniformly bounded on K, and since the latter are all harmonic functions in K, the uniform convergence of (13) on K can easily be seen by using Poisson integrals and Lebegue's Bounded Convergence Theorem. The next step is to prove (14). Let $x \in \mathrm{supp}(d\alpha) \setminus [-1,1]$, and let the measure $d\nu = d\nu_x$ be defined by $d\nu(E) = d\alpha(E \setminus \{x\})$ for every Borel set E. Notice that $d\nu$ also satisfies the conditions of Theorem 1. By formula (7.9) in [10, p. 132]

$$(19) \qquad p_n(d\alpha,x) = \frac{\gamma_n(d\nu)}{\gamma_n(d\alpha)}\, p_n(d\nu,x)[1 + \varepsilon\lambda_{n+1}^{-1}(d\nu,x)]^{-1}$$

where $\varepsilon = \alpha(x+0) - \alpha(x-0) = d\alpha(\{x\})$. By simple computation we obtain from (13) the limit relation

$$(20) \qquad \lim_{n\to\infty} \lambda_{n+1}(d\nu,x)^{-1/n} = \lim_{n\to\infty} [\sum_{k=0}^{n} p_k^2(d\nu,x)]^{1/n} = |x + \sqrt{x^2-1}\,|^2 ,$$

and thus (14) follows from (11), (19) and (20). Now we turn to proving (15). Let K be a compact subset of G. Then there exists a simply connected domain G_1 such that $K \subset G_1$ and $\overline{G}_1 \subset G$. While proving (13) we showed that there exists $N = N(G_1)$ such that for $n \geq N$ $p_n(d\alpha) \neq 0$ in G_1. Thus $x + \sqrt{x^2-1}$ and $p_n(d\alpha,x)^{1/n}$, $n \geq N$, are defined and analytic in G_1, and by (13)

$$\lim_{n\to\infty} |p_n(d\alpha,x)^{1/n}/(x + \sqrt{x^2-1})| = 1 .$$

We may assume without loss of generality that G_1 contains a real number x_0 greater than any number in support of $d\alpha$, and then by (13) we have

$$\lim_{n\to\infty} p_n(d\alpha,x_o)^{1/n}/(x_o + \sqrt{x_o^2 - 1}) = 1 .$$

Now (16) follows from Vitali's theorem about convergent subsequences of bounded sequences of analytic functions, or from Problem 11.18 in [13, p. 239] applied to $\log(p_n(d\alpha,x)^{1/n}/(x + \sqrt{x^2 - 1}))$. Theorem 2 has completely been proved.

Remark 2. Formulas (10)-(15) generalize Erdös-Turán's celebrated results on distribution of zeros of orthogonal polynomials (see e.g. [2, Ch. 3.7 and 3.9]), and they can be used to prove theorems on convergence of orthogonal Fourier series and Lagrange interpolating processes associated with analytic functions.

While proving (13) we showed that if $x \in \mathrm{supp}(d\alpha)\setminus[-1,1]$ then for every small enough $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that every polynomial $p_n(d\alpha)$, $n \geq N$, has exactly one zero in $(x-\varepsilon,x+\varepsilon)$. Denoting this zero of $p_n(d\alpha)$ by $r_n(d\alpha,x)$, we have the following

Corollary. Let $d\alpha$ satisfy the conditions of Theorem 1 and let $x \in \mathrm{supp}(d\alpha)\setminus[-1,1]$ be fixed. Then

$$\lim_{n\to\infty}|x - r_n(d\alpha,x)|^{1/n} = |x + \sqrt{x^2 - 1}\,|^{-2} . \tag{21}$$

Proof. Let us define Q_n by

$$p_n(d\alpha,t) = (t - r_n(d\alpha,x))Q_n(t) . \tag{22}$$

Then the proof of (13) applied to Q_n yields

$$\lim_{n\to\infty}|Q_n(x)|^{1/n} = |x + \sqrt{x^2 - 1}\,|$$

and therefore (21) follows from (14) and (22).

The upcoming lemma was used in the proof of Theorem 1. In what follows we define $K_n(d\alpha,x,t)$ by

$$K_n(d\alpha,x,t) = \sum_{k=0}^{n-1} p_k(d\alpha,x)p_k(d\alpha,t) .$$

Lemma. Let $d\alpha$ be such that for every $\varepsilon > 1$ the set $\operatorname{supp}(d\alpha)\setminus[-\varepsilon,\varepsilon]$ is finite. Then for every nonnegative continuous function f vanishing at ± 1 the inequality

$$\limsup_{n\to\infty} \int_{-1}^{1} n\lambda_n(d\alpha,x)f(x)dx \leq \pi \int_{-1}^{1} f(x)\sqrt{1-x^2}\, d\alpha(x) \tag{23}$$

holds.

Proof. Fix $\varepsilon > 1$ and choose a polynomial R so that $R(x) = 0$ if and only if $x \in \operatorname{supp}(d\alpha)\setminus[-\varepsilon,\varepsilon]$. Let m denote the degree of R and let $d\beta$ be the Chebyshev measure in $[-1,1]$, that is $\operatorname{supp}(d\beta) = [-1,1]$ and $d\beta(x) = (1-x^2)^{-1/2}\,dx$ for $-1 \leq x \leq 1$. Then by the extremal property (3) we have

$$\lambda_n(d\alpha,x) \leq R^{-2}(x)\, \lambda_{n-m}^2(d\beta,x/\varepsilon) \int_{-\varepsilon}^{\varepsilon} R^2(t)\, K_{n-m}^2(d\beta,x/\varepsilon,t/\varepsilon)d\alpha(t) .$$

First multiplying this inequality by $nf(x)$ and integrating the resulting inequality between -1 and 1, and then interchanging the order of integration on the right side and replacing the variables x/ε and t/ε by x and t respectively, we obtain

$$\int_{-1}^{1} n\lambda_n(d\alpha,x)f(x)dx \leq I_n \tag{24}$$

where

$$I_n = \varepsilon \int_{-1}^{1} R^2(\varepsilon t) \int_{-1/\varepsilon}^{1/\varepsilon} n\lambda_{n-m}^2(d\beta,x)R^{-2}(\varepsilon x)f(\varepsilon x)\, K_{n-m}^2(d\beta,x,t)dx\, d\alpha(\varepsilon t) . \tag{25}$$

All we have to do is to prove

$$(26) \qquad \lim_{n\to\infty} I_n = \pi \int_{-1}^{1} f(t) \sqrt{\varepsilon^2 - t^2}\, d\alpha(t)$$

and then (23) follows from (24) by letting ε tend to 1. It is well known [10, p. 79] that

$$(27) \qquad \lambda_n(d\beta,x) = \frac{\pi}{n} + O(n^{-2}) , \qquad n = 1, 2, \ldots ,$$

holds uniformly on every closed subinterval in $(-1,1)$ and

$$(28) \qquad \int_{-1}^{1} K_n^2(d\beta,x,t)(1 - x^2)^{-1/2}\, dx = K_n(d\beta,x,x) \leq \frac{2n-1}{\pi}$$

for $n = 1, 2, \ldots$ and $-1 \leq x \leq 1$. Hence we can rewrite (25) as

$$(29) \qquad I_n = \pi\varepsilon \int_{-1}^{1} R^2(\varepsilon t) \frac{\pi}{n} \int_{-1/\varepsilon}^{1/\varepsilon} R^{-2}(\varepsilon x) f(\varepsilon x) K_{n-m}^2(d\beta,x,t)dx\, d\alpha(\varepsilon t) + O(\tfrac{1}{n}) ,$$

$n = 1, 2, \ldots$. It follows from (27), (28) and Theorem 6.2.2 in [10, p. 62] that

$$(30) \qquad \lim_{n\to\infty} \frac{\pi}{n} \int_{-1/\varepsilon}^{1/\varepsilon} R^{-2}(\varepsilon x) f(\varepsilon x) K_{n-m}^2(d\beta,x,t)dx = \begin{cases} 0 & , \ 1/\varepsilon < |t| \leq 1 \\ R^{-2}(\varepsilon t) f(\varepsilon t) \sqrt{1-t^2} & , \ |t| \leq 1/\varepsilon \end{cases}$$

and

$$0 \leq \frac{\pi}{n} \int_{-1/\varepsilon}^{1/\varepsilon} R^{-2}(\varepsilon x) f(\varepsilon x) K_{n-m}^2(d\beta,x,t)dx \leq C , \qquad n = 1, 2, \ldots ,$$

where C does not depend on n. Thus (26) follows from (29) via Lebegue's Bounded Convergence Theorem.

Let us point out that (30) remains valid even if $f(\pm 1) \neq 0$. However, in this case (30) does not hold for $|t| = 1/\varepsilon$ and thus our proof does not yield (26) if α is not continuous at ± 1. Fortunately, this problem may be overcome by showing that the limit superior of the expression in (30) for $|t| = 1/\varepsilon$ is of order $\sqrt{1 - \varepsilon^{-2}}$ when $\varepsilon \downarrow 1$. Since we do not need this refinement of the Lemma in this paper we spare the reader from the details.

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