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L^p - BEHAVIOUR OF POWER SERIES WITH POSITIVE COEFFICIENTS AND SOME SPACES OF ANALYTIC FUNCTIONS

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In [2], [3] and [4] we have considered some theorems on L^p-behaviour of power series with positive coefficients and their applications to Hardy spaces and generalized Lipschitz spaces of analytic functions. Now, we extend some of the results obtained there to a wider class of spaces.

Let $F : [0,1] \times [0,+\infty[$ be a continuous function satisfying

(1) $$C^{-1}(xy)^{\beta} F(r,t) \leq F(xr,yt) \leq C\, y^{\alpha} F(r,t)$$

for all $x,y,r \in [0,1]$, $t \in [0,\infty[$, where $C, \alpha, \beta \in [0,\infty[$ do not depend on x, y, r, t.

Throughout the paper the letter A denotes a positive absolute constant and need not be the same on each occurence.

THEOREM 1. Let $a_n \geq 0$ $(n \geq 0)$. Then

(2) $$\int_0^1 F(1-r, \sum_{n=0}^{\infty} a_n r^n) \frac{dr}{1-r} < \infty.$$

if and only if

(3) $$\sum_{n=0}^{\infty} \frac{1}{n+1} F(\frac{1}{n+1}, \sum_{k=0}^{n} a_k) < \infty .$$

Proof. Let $s_n = a_0+a_1+\dots+a_n$, $\alpha_n = 2^{n-1}s_{2^n}$ $(n>0)$, $\alpha_0 = s_0$ $\varphi(r) = F(1-r, \sum a_n r^n)$. We observe

(4) $$\sum_{n=0}^{\infty} a_n r^n = (1-r) \sum_{n=0}^{\infty} s_n r^n$$

By the condesation,

$$(5) \qquad \sum_{n=0}^{\infty} s_n r^n \le \alpha_0 + \sum_{n=0}^{\infty} \alpha_{n+1} r^{2^n} .$$

Let $\eta_{n+1} = 2^{n\delta} r^{2^{n-1}}$, $\theta_{n+1} = 2^{-n\delta} \alpha_{n+1} r^{2^{n-1}}$ $(n \ge 0)$, $\theta_0 = s_0$ and $0 < \delta < 1$. Because

$$\sum_{n=0}^{\infty} 2^{n\gamma} r^{2^n} \le Ar(1-r)^{-\gamma}, \quad \gamma > 0,$$

we find

$$(6) \qquad \alpha_0 + \sum_{n=0}^{\infty} \alpha_{n+1} r^{2^n} \le A(\sup_{n\ge 0} \theta_n) \ (1-r)^{-\delta}$$

From (4), (5) and (6), it follows

$$(7) \qquad \varphi(r) \le K\ F[1-r, (\sup_{n\ge 0} \theta_n)\ (1-r)^{1-\delta}] \ .$$

by (1). Here and elsewere K denotes a positive constant which depends only on C, α and β and need not be the same in each occurence. Since F is a continuous function, we have

$$\varphi(r) \le K \sum_{n=0}^{\infty} c_n(r) \ ,$$

by (7), where

$$c_n(r) = F[1-r,\ \theta_n(1-r)^{1-\delta}] \quad (n \ge 0).$$

Changing the variable r using $1-r = t/2^n$ $(n > 0)$, we have

$$c_{n+1} = \int_0^1 c_{n+1}(r)dr = \int_0^{2^n} \varphi_n(t)dt \quad ,$$

where $\varphi_n(t) = F(t/2^n,\ 2(1-t/2^n)^{2^{n-1}} \alpha_{n+1}\ t^{1-\delta}\ 2^{-n})$.

By (1),

$$\varphi_n(t) \le \begin{cases} K\ F(2^{-n},\ s_{2^{n+1}})\ \ t^{\alpha(1-\delta)} & , 0 \le t \le 1 \\ K\ F(2^{-n},\ s_{2^{n+1}})\ (t)^{\beta}\ t^{(1-\delta)\beta}\ e^{-\alpha t/2}, & 1 \le t \le 2^n \end{cases}$$

From this estimate and the left hand side of (1), it follows

$$c_{n+1} \le K\ F(2^{-n-1},\ s_{2^{n+1}}) \quad (n \ge 0)$$

Hence (3) implies (2) by (1) and a Cauchy condensation test type argument.

Converselv, suppose that (2) holds. Let $r_n = 1 - \frac{1}{n}$ $(n \ge 1)$, then

$$(8)\qquad \sum_{n=1}^{\infty} n \int_{r_n}^{r_{n+1}} \varphi(r)\,dr < +\infty .$$

By (1),

$$(9)\qquad \varphi(r) \geq C^{-1} F\left(\frac{1}{n+1} \cdot \sum_{k=0}^{n} a_k\, r_n^k\right) \geq K\, F\left(\frac{1}{n+1}, s_n\right), \quad r'_n \leq r \leq r_{n+1}$$

Thus (2) implies (3), by (8) and (9).

Throughout the paper let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the open unit disc.

We use the usual notations

$$\sigma_n f(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k z^k ,$$

$$M_p^p(r,f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p\, dt, \quad 0 < p < +\infty ,$$

$$M_\infty(r,f) = \sup_t |f(re^{it})|$$

and we write

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r,f) .$$

THEOREM 2. Let $1 \leq p \leq +\infty$. Then

$$(10)\qquad \int_0^1 F\left[1-r,\ M_p(r,f)\right] \frac{dr}{1-r}$$

if and only if

$$(11)\qquad \sum_{n=0}^{\infty} \frac{1}{n+1} F\left(\frac{1}{n+1}, \|\sigma_n f\|_p\right) < \infty$$

Proof. Let $x_n = \sum_{k=0}^{n} (k+1)(n-k+1) \|\sigma_k\|_p$. Since (cf. [3])

$$x_n \leq A(n+1)^3 \|\sigma_n\|_p ,$$

we find that (11) implies

$$(12)\qquad \sum_{n=0}^{\infty} \frac{1}{n+1} F\left(\frac{1}{n+1}, \frac{x_n}{(n+1)^3}\right) < \infty .$$

Since (x_n) is a monotonically increasing sequence, from (1) and (12) it follows

$$(13)\qquad \sum F\left(2^{-n}, 2^{-4n} \sum_{k \in I_n} x_k\right) < \infty,$$

by a Cauchy condensation test type argument, where $I_n = \{k : 2^{n-1} \leq k < 2^n\}$ $(n > 0)$, $I_0 = \{0\}$. Since (cf. [3])

$$M_p(r,f) \leq (1-r)^4 \sum_{n=0}^{\infty} x_n r^n ,$$

we conclude that (13) implies (10) in a similar way as in the proof of Theorem 1.

Conversely, suppose that (10) holds. Let $\varphi(r)=F[1-r,M_p(r,f)]$ and $r_n = 1-\frac{1}{n}$ $(n\geq 1)$. Then (10) implies

$$(14) \qquad \sum_{n=1}^{\infty} n\int_{r_n}^{r_{n+1}} \varphi(r)dr < +\infty .$$

Since (cf. [3]) $M_p(r_n,f)\geq r_n^n \|\sigma_n f\|_p \geq A\|\sigma_n f\|_p$ $(n\geq 1)$, from (1) it follows

$$\int_{r_n}^{r_{n+1}} \varphi(r)dr \geq \frac{K}{(n+1)^2} F\left(\frac{1}{n+1}, \|\sigma_n f\|_p\right).$$

Hence (14) implies (11).

We say that function F is normal if it satisfies (1) and

$$(15) \qquad F(xr,yt)\leq C(xy)^{\alpha} F(r,t), \qquad 0\leq x,y\leq 1 .$$

We state the following results without proofs.

THEOREM 3. Let F be a normal function and (a_n) $(n\geq 0)$ non-negative sequence of real numbers. Then the following conditions are equivalent: (2) and

$$\sum_{n=0}^{\infty} F(2^{-n}, \sum_{k\in I_n} a_k) < \infty .$$

We define $\Delta_n f(z) = \sum_{k\in I_n} a_k z^k$ $(n\geq 0)$.

THEOREM 4. If F is a normal and $1<p<\infty$, then the condition (10) is equivalent to

$$\sum_{n=0}^{\infty} F\left[2^{-n}, \|\Delta_n f\|_p\right] < \infty .$$

The kernels w_n (with the coefficients $\hat{w}_n(k)$) are defined in the following way. If $n>0$, then $\hat{w}_n(2^n)=1$, $\hat{w}_n=0$ outside $(2^{n-1},2^{n+1})$ and $\hat{w}_n$ is a linear function of the intervals $[2^{n-1},2^n]$ and $[2^n,2^{n+1}]$; $w_0(z) = 1+z$.

THEOREM 5. If F is a normal function and $1\leq p\leq\infty$, then the condition (10) is equivalent to

$$\sum_{n=0}^{\infty} F\left[2^{-n}, \|w_n * f\|_p\right] < \infty .$$

where $(w_n * f)(z) = \sum w_n(k) a_k z^k$.

THEOREM 6. Let F be a normal function, $1 < p < +\infty$, and (a_n) a monotonically decreasing sequence. Then (10) is equivalent to

$$\sum_{n=0}^{\infty} \frac{1}{n+1} F\left[\frac{1}{n+1} \cdot (n+1)^{1-1/p} a_n\right] < \infty \quad .$$

If $F(r,t) = r^{\alpha} \theta(t)$ or $F(r.t) = \varphi(r) t^q$, $r \in [0,1]$, $t \in [0,\infty[$, $(\alpha, q > 0)$ (for the definit ion functions $\varphi(r)$ and $\theta(t)$ see [2] and [3]), we observe that results presented here include the corresponding results obtained in [2], [3] and [4].

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