КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 605—609
CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.605-609

ON AN EXTREMAL PROBLEM IN THEORY OF APPROXIMATION

Igor Ž. Milovanović

As it was said, monography [1], the following problem often appears during examination of approximative properties of positive polynomial operators: In the set of all nonnegative trigonometric polynomial of order n

$$x(t) = 1 + 2q_1 \cos t + \dots + q_n \cos nt$$

the one whose coefficient q_1 is the biggest should be found. As it should be seen this problem is reduced on solving of extremal problem

(1) $$\sum_{k=o}^{n-1} x_k x_{k+1} \to \sup \; ; \; \sum_{k=o}^{n} x_k^2 = 1.$$

The solution of this problem (see [1]) is $q_1 = \cos \frac{\pi}{n+2}$.

Similar problem

(2) $$\sum_{k=1}^{n-1} (x_k - x_{k+1})^2 \to \sup \; ; \; \sum_{k=1}^{n} x_k^2 = 1 ,$$

could be found in matrix theory (see for example [2]).

A great number of papers (see for example [3],...,[8]) are concerned with solving problems (1) and (2). These problems are also solved in integral form.

Using the method from [3], in this paper more general problem than (2) will be solved (by the same procedure a generalized problem (1) is solved).

Let $p=(p_1,\dots,p_{n+1})$ and $r=(r_1,\dots,r_n)$ be two given weight sequences. For a sequence $x=(x_1,\dots,x_n)$ let us define quantites

(3) $$||x||_p^2 = \sum_{k=1}^{n} p_k x_k^2$$

and

(4) $$(\Delta x,\Delta x)_r = \sum_{k=1}^{n-1} r_k(x_k-x_{k+1})^2,$$

where the sequence Δx is given by

$$\Delta x = (x_1-x_2,\ldots,x_{n-1}-x_n).$$

In this paper, we will consider the determination of the best constants A_n and B_n in the inequalities

(5) $$A_n||x||_p^2 \leqq (\Delta x,\ \Delta x)_r \leqq B_n||x||_p^2$$

under condition

(6) $$\sum_{i=1}^{n} \sqrt{p_i}x_i = 0\ .$$

Theorem. Define a sequence of polinomials $(Q_x(x))$ for the given weight sequence r and p using the recursive relation

(7) $$\begin{aligned} &xQ_{k-1}(x) = b_kQ_k(x) + a_kQ_{k-1}(x) + b_{k-1}Q_{k-2}(x) \\ &Q_{-1}(x) = 0,\quad Q_o(x) = 1\ , \end{aligned}$$

where

(8) $$a_k = \frac{r_{k-1}+r_k}{p_k}\quad (r_o=0),\ k=1,\ldots,n,\quad b_k = \frac{-r_k}{\sqrt{p_kp_{k+1}}}\ ,\quad k=1,\ldots,n-1.$$

For each sequence of real numbers $x=(x_1,\ldots,x_n)$ with the property (6), the inequality (5) holds, where A_n and B_n are minimum and maximum zeros ($\neq 0$) of polynomial $x \to R_n(x)$ given by

$$R_n(x) = \frac{r_n}{\sqrt{p_n}}\left(\frac{1}{\sqrt{p_n}}Q_{n-1}(x) - \frac{1}{\sqrt{p_{n+1}}}Q_n(x)\right)\ .$$

The equality in the left (right) inequality (5) holds if and only if

$$x_k = \frac{C}{\sqrt{p_k}}Q_{k-1}(\lambda)\qquad (k=1,\ldots,n),$$

where $\lambda=A_n(\lambda=B_n)$ and C is an arbitrary real constant different from zero.

Proof. Let X be Euklid´s space of n-dimensional vector with scalar product $(\vec{z},\vec{w}) = \sum_{k=1}^{n} z_k w_k$, where $\vec{z} = [z_1 \cdots z_n]^T$ and $\vec{w} = [w_1 \cdots w_n]^T$.

Let, further $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_{n-1})$ and three-diagonal matrix

$$H_n(a,b) = \begin{bmatrix} a_1 & b_1 & 0 & \ldots & 0 & 0 \\ b_1 & a_2 & b_2 & & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & & b_{n-1} & a_n \end{bmatrix}$$

Introducing $z_k = \sqrt{p_k}x_k$ $(k=1,\ldots,n)$ (then $\sum_{k=1}^{n} z_k=0$), the quantites (3) and (4) become

$$||x||_r^2 = \sum_{k=1}^{n} r_k x_k^2 = \sum_{k=1}^{n} z_k^2 = (\vec{z},\vec{z})$$

and

$$(\Delta x,\Delta x) = \sum_{k=1}^{n-1} \frac{r_k}{\sqrt{p_k p_{k+1}}}(\sqrt{p_{k+1}}\, z_k - \sqrt{p_k}\, z_{k+1})^2 = (H_n(a,b)\vec{z},\vec{z}).$$

On the other, let us consider a sequence of polynomial $(Q_k(x))$ wich is defined by (7). For k=1,...,n, on the basis of (7), we obtain the equality

(9) $$x\vec{v} = H_n(a,b)\vec{v} + \frac{r_n}{\sqrt{p_n}} R_n(x)\, \vec{e}$$

where

$$\vec{v} = [Q_o(x)Q_1(x)\cdots Q_{n-1}(x)]^T \quad \text{and} \quad \vec{e} = [0\cdots 01]^T.$$

Now for $x=\lambda$ the conclusions follow from equality (9). If λ is a zero polynomial $R_n(x)$, it is also eigenvalue of matrix $H_n(a,b)$, and if λ is eigenvalue of matrix $H_n(a,b)$ then λ must be zero of polynomial $R_n(x)$.

Thus, eigenvalues of matrix $H_n(a,b)$ are zeros of polynomial $R_n(x)$ in the same time. Since $H_n(a,b)$ is a three-diagonal matrix $(b_k^2 > 0,\ k=1,\ldots,n-1)$ all its eigenvalues $\lambda_k(k=1,\ldots,n-1)(\lambda_n=0)$ are real and distinct

$$A_n(\vec{z},\vec{z}) \le (H_n(a,b)\vec{z},\vec{z}) \le B_n(\vec{z},\vec{z})$$

hold, with equality for eigenvectors corresponding to eugenvalues $A_n = \min_k \lambda_k$ and $B_n = \max_k \lambda_k$, $k=1,\dots,n-1$.

Corollary 1. For any sequence of real numbers $x=(x_1,\dots,x_n)$, with property $\sum_{k=1}^{n} x_k=0$, the inequality

$$\sum_{k=1}^{n-1} k(x_k-x_{k+1})^2 \le B_n \sum_{k=1}^{n} x_k^2$$

is valid. B_n is a maximal zero of the generalized Laguerre polynomial $x \to L_{n-1}^{(-1)}(x)$.

Corollary 2. For any sequence of real numbers $x=(x_1,\dots,x_n)$, with property $\sum_{k=1}^{n} x_k=0$, the following inequalities are valid

$$(10) \qquad 4\sin^2 \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2 \le \sum_{k=1}^{n-1} (x_k-x_{k+1})^2 \le 4\cos^2 \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2 .$$

The equality in the left inequality (10) holds if and only if

$$x_k = A \cos \frac{2k-1}{2n} \pi, \quad k=1,\dots,n,$$

where A=const.

The equality in the right inequality (10) holds if and only if

$$x_k = (-1)^{k-1} A \sin \frac{2k-1}{2n} \pi, \quad k=1,\dots,n,$$

where A=const.

It should be noted that the method used in this paper has the possibility of obtaining two fold inequalities. This was not the case in referenced papers.

References

1. V.M.Tihomirov. Nekotorie voprosi teorii približenii. MGU, Moskva, 1976.
2. R.Belman. Introduction to matrix analysis. McGraw-Hill, New York, 1960.
3. G.V.Milovanović, I.Ž.Milovanović. On discrete inequalities of Wirtinger´s type. J.Math.Anal.Appl. 88(1982), 378-387.

4. K.Fan, O.Taussky, J.Todd. Discrete analogs of inequalities of Wirtinger. Monatsh. Math. 59(1955), 73-90.

5. N.D.Block. Discrete an analogues of certain integral inequalities. Proc. Amer.Math.Soc. 8, No 4(1957), 852-859.

6. O.Shisha. On the discrete version of Wirtinger´s inequality. Amer.Math. Monthly. (1973), 755-760.

7. J.Novotna. Variations of discrete analogues of Wirtinger´s inequality. Čas.pešt.mat. 105(1980), 278-285.

8. J.Novotna. A shorpening of discrete analogues of Wirtinger´s inequality. Čas.pešt.mat. 108(1983), 70-77.

9. D.S.Mitrinović, P.M.Vasić. Analytic Inequalities. Springer-Verlag, Berlin-New York-Heidelberg, 1970.

10. H.Bateman, A.Erdelyi. Higher Transcendental Functions. McGraw-Hill, New York, 1953.

Department of Mathematics
Faculty of Electronic Engeneering
P.O.Box 73, 18000 Niš, Yugoslavia