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CONVEX AND MONOTONE SPLINE INTERPOLATION

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1. Introduction. In the practical use of spline interpolation it is often required to preserve such properties of the data as monotonicity and convexity. For example, this must be done when the splines are used for describing the geometric shapes. It is very important to preserve monotonicity or convexity in the interpolation of functions which have the neighbouring pieces with large and small gradients. Certainly, the preserving of monotonicity and convexity is very useful for numerical differentiation.

Let in the knots of a mesh $\Delta : a = x_0 < x_1 < \dots < x_N = b$ the data f_i , $i = 0, \dots, N$, be given.

The data $\{f_i\}$ are called monotone provided

$$f_0 \le f_1 \le \dots \le f_N \ , \quad (f[x_{k-1}, x_k] \ge 0, \ k = 1, \dots, N). \qquad (1)$$

The data $\{f_i\}$ are called convex provided

$$f[x_{k-1}, x_k, x_{k+1}] \ge 0 \ , \quad k = 1, \dots, N-1 . \qquad (2)$$

By the symbols $f[x_{k-1}, x_k]$, $f[x_{k-1}, x_k, x_{k+1}]$ we denote the first and second order divided differences.

The problem of monotone interpolation consists in the construction of the interpolating spline $S(x): S(x_i) = f_i$, $i = 0, \dots, N$ such that $S'(x) \ge 0$, $x \in [a, b]$, provided the data $\{f_i\}$ are monotone (Fig. 1). If the data $\{f_i\}$ are convex, it is required to construct convex spline $S(x)$, that is $S''(x) \ge 0$, $x \in [a, b]$ (Fig. 2). Of course, the existence of S' or S'' is supposed.

It is well known that the classical interpolating splines [1-3], for example, the cubic or parabolic spline, in general, do not preserve monotonicity and convexity of the data (Fig. 3). Therefore, to construct monotone or convex interpolation, special methods are needed. The characteristic feature of these methods is the presence of free parameters in the formula of a spline, which are used to control the

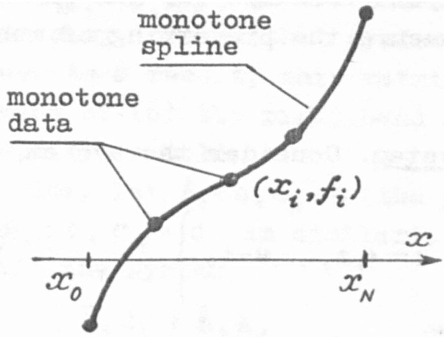

Fig. 1.

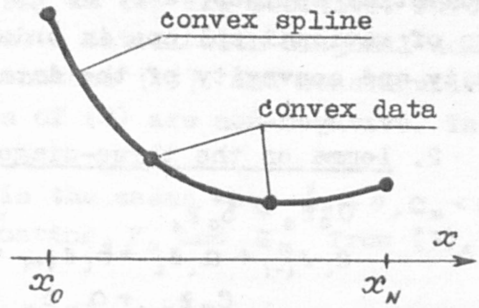

Fig. 2.

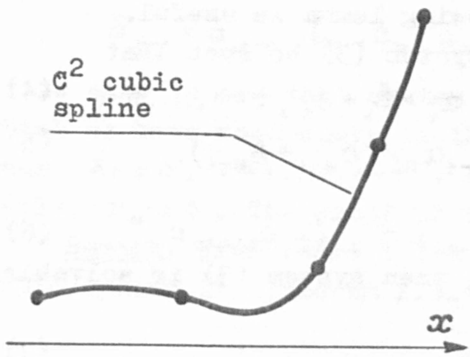

Fig. 3.

the form of a spline. The free parameters appear if either splines of high degree [4,5], or splines with additional knots [3,6-10], or else non-polynomial splines (exponential, rationa, etc.) [2,9-13] are applied.

Let $C^k[a,b]$ (C^k) be a class of k-times continuously differentiable functions. In this paper we discuss the monotone and convex interpolation by splines $S(x) \in C^2$. In practice, the C^2 cubic splines are very often used. We give sufficient conditions of monotonicity (convexity) for these splines, provided the interpolated data are monotone (convex). The conditions are formulated in terms of divided differences over Δ and they are very easy for testing. The method that was used to deduce these conditions is based on the simple lemma about the threediagonal system. It may be applied to any spline if the construction of the spline is reduced to solution of threediagonal system with diagonal dominance. For example, we give sufficient conditions of monotonicity and convexity for the parabolic spline.

If the cubic spline does not preserve monotonicity or convexity of the data, we suppose to use rational splines [10,11]. They generalize Späth's rational spline [14]. The name "rational" reflects only the form that we use to write the formula of a spline. In fact, these splines include, as special cases, the C^2 cubic spline, the cubic spline with additional knots, various types or rational splines, the exponential splines, etc. We give explicit formulas for the parameters of rational splines in order to secure the preserving of monotonicity and convexity of the data.

2. Lemma on the three-diagonal system. Consider the system

$$\left.\begin{aligned} a_0 z_0 + b_0 z_1 &= d_0, \\ c_i z_{i-1} + a_i z_i + b_i z_{i+1} &= d_i, \quad i = 1, 2, \ldots, N-1, \\ c_N z_{N-1} + a_N z_N &= d_N. \end{aligned}\right\} \tag{3}$$

Let the system be solvable and its right-hand members be positive. What can we say on the sign of the values z_i ? It is known that $z_i \geq 0$, $i = 0, \ldots, N$, if the matrix of system (3) is monotone [15]. But the matrices that occur by the construction of splines are usually not monotone. In this case the following lemma is useful.

Lemma 1. Let the coefficients of system (3) be such that

$$a_i > 0, \; i = 0, \ldots, N; \quad c_i \geq 0, \; b_i \geq 0, \; a_i > b_i + c_i, \; i = 1, \ldots, N-1, \tag{4}$$

$$b_0 < a_0 a_1/(c_1 + b_1), \quad c_N < a_{N-1} a_N/(c_{N-1} + b_{N-1}). \tag{5}$$

If

$$d_i \geq 0, \quad d_i - b_i d_{i+1}/a_{i+1} - c_i d_{i-1}/a_{i-1} \geq 0, \quad i = 0, \ldots, N \tag{6}$$

(here $c_0 = b_N = d_{-1} = d_{N+1} = 0$, $a_{-1} = a_{N+1} = 1$), then system (3) is solvable and

$$z_i \geq 0, \quad i = 0, \ldots, N. \tag{7}$$

Proof. First we consider the case $b_0 \geq 0, c_N \geq 0$. We add to (3) equations $a_i z_i = d_i$, with $a_i = 1, d_i = 0, i = -1, N+1$ and suppose $c_0 = c_{N+1} = b_{-1} = b_{N+1} = 0$. For each $i = 0, \ldots, N$ we take the linear combination of the (i-1)th, i-th and (i+1)th equations of this system with corresponding coefficients $-c_i/a_{i-1}$, 1, $-b_i/a_{i+1}$. Then we have

$$\left.\begin{aligned} A_0 z_0 - B_0 z_2 &= D_0, \\ -C_i z_{i-2} + A_i z_i - B_i z_{i+2} &= D_i, \quad i = 1, 2, \ldots, N-1, \\ -C_N z_{N-2} + A_N z_N &= D_N, \end{aligned}\right\} \tag{8}$$

where $C_i = c_{i-1} c_i / a_{i-1}$, $A_i = a_i - c_i b_{i-1}/a_{i-1} - c_{i+1} b_i / a_{i+1}$,

$B_i = b_i b_{i+1}/a_{i+1}$, $D_i = d_i - c_i d_{i-1}/a_{i-1} - b_i d_{i+1}/a_{i+1}$,

and evidently $C_1 = B_{N-1} = 0$. Thus, (8) is the system with unknowns $z_0, \ldots, z_N$. By using (4) we have $C_i \geq 0$, $B_i \geq 0$, $i = 0, \ldots, N$ and $A_i \geq a_i - b_i - c_i > 0$, $i = 2, \ldots, N-2$. Next, by using (5), we obtain

$A_1 > a_1 - a_1 c_1/(c_1 + b_1) - b_1 c_2 / a_2 = b_1 \{ a_1/(c_1 + b_1) - c_2/a_2 \} > 0,$

$A_0 = a_0 - c_1 b_0 / a_1 > a_0 - c_1 a_0/(c_1 + b_1) > 0,$

and $A_{N-1}>0, A_N>0$. Thus, all A_i in (8) are positive. Further, it is easy to show that system (8) has a matrix with diagonal dominance. As a result, this matrix is monotone [15] and nonsingular. In view of (6) the right-hand members of (8) are non-negative. Therefore, (7) is hold.

Now, let $b_0<0, c_N<0$ (the proof in the cases when $b_0 \geq 0, c_N<0$ or $b_0<0,\ c_N \geq 0$ is similar). Eliminating z_0 and z_N from (3), we obtain the system

$$\left.\begin{aligned} &\tilde{a}_1 z_1 + b_1 z_2 = \tilde{d}_1, \\ &c_i z_{i-1} + a_i z_i + b_i z_{i+1} = d_i,\quad i=2,\ldots,N-2, \\ &c_{N-1} z_{N-2} + \tilde{a}_{N-1} z_{N-1} = \tilde{d}_{N-1}, \end{aligned}\right\} \tag{9}$$

where $\tilde{a}_1 = a_1 - c_1 b_0/a_0$, $\tilde{d}_1 = d_1 - c_1 d_0/a_0$,

$$\tilde{a}_{N-1} = a_{N-1} - b_{N-1} c_N/a_N\ ,\ \tilde{d}_{N-1} = d_{N-1} - b_{N-1} d_N/a_N .$$

It is easy to see that system (9) satisfies all of the hypotheses which we have used above in the study of the case $b_0 \geq 0, c_0 \geq 0$. Thus, $z_i \geq 0$, $i=1,\ldots,N-1$. But $z_0=(d_0-b_0 z_1)/a_1>0$ and, similarly, $z_N \geq 0$. The proof of the lemma is finished.

<u>Remark</u>. From Lemma 1 the general existence theorem follows for cubic spline [1, theorem 2.9.1] in non-periodic case.

3. <u>Convex cubic spline</u>. Let $S(x)$ be a cubic interpolating spline, that is: $S(x)$ - cubic polynomial on each interval $[x_i, x_{i+1}]$; $S(x_i)=f_i$, $i=0,\ldots,N$ and $S(x) \in C^2[a,b]$. We denote $M_i = S''(x_i)$, $i=0,\ldots,N$; $h_i = x_{i+1}-x_i$, $i=0,\ldots,N-1$; $\lambda_i = h_i/(h_{i-1}+h_i)$, $\mu_i = 1-\lambda_i$, $i=1,\ldots,N-1$. The end conditions for the spline we take in the form

$$2M_0 + \lambda_0 M_1 = d_0\ ,\quad \mu_N M_{N-1} + 2M_N = d_N, \tag{10}$$

where the values $\lambda_0, \mu_N, d_0, d_N$ are given.

<u>Theorem 1</u>. Suppose $S(x)$ is a cubic spline with end conditions (10). Let $S(x)$ interpolate the convex data $\{f_i\}$. If

$$\begin{gathered} \lambda_0<4\ ,\ \mu_N<4\ ,\ d_0 \geq 0\ ,\ d_N \geq 0\ , \\ 2d_i - \lambda_i d_{i+1} - \mu_i d_{i-1} \geq 0,\quad i=0,1,\ldots,N, \end{gathered} \tag{11}$$

where $d_i = 6f[x_{i-1}, x_i, x_{i+1}]$, $i=1,\ldots,N-1$; $d_{-1} = d_{N+1} = 0$, then $S(x)$ is convex on $[a,b]$, that is $S''(x) \geq 0$, $x \in [a,b]$.

<u>Proof</u>. On each interval $[x_i, x_{i+1}]$ we have

$$S''(x) = (1-t)M_i + tM_{i+1},\quad t=(x-x_i)/h_i .$$

Therefore, to prove the theorem it suffices to show that $M_j \geq 0, j=0,\ldots,N$ This assertion we obtain from the system **for moments** M_j , $j=0,\ldots,N$

(see [1,2]) by applying Lemma 1.

Remark. For the first time the result of the theorem was formulated in [10] (with stronger requirements on $\lambda_0, \mu_N : 0 \le \lambda_0, \mu_N < 4$).

Define $H = \max_i h_i$. If we interpolate the function $f(x) \in C^2$, $f''(x) > 0$, $x \in [a,b]$, then conditions (11) would be fulfilled for any mesh Δ provided H is sufficiently small.

4. Monotone cubic spline. Let $m_i = S'(x_i)$, $i = 0,\dots,N$. We take the boundary conditions for the spline in the form

$$2m_0 + \mu_0 m_1 = \tilde{c}_0 , \quad \lambda_N m_{N-1} + 2m_N = \tilde{c}_N , \tag{12}$$

Here $\mu_0, \lambda_N, \tilde{c}_0, \tilde{c}_N$ are the given numbers.

Lemma 2. Suppose $S(x)$ is a cubic spline with end conditions (12). Let $S(x)$ interpolate the monotone data $\{f_i\}$. If

$$\mu_0 < 4 , \quad \lambda_N < 4 , \quad \tilde{c}_0 \ge 0 , \quad \tilde{c}_N \ge 0 ,$$

$$2\tilde{c}_i - \mu_i \tilde{c}_{i+1} - \lambda_i c_{i-1} \ge 0 , \quad i = 0,1,\dots,N ,$$

where $\tilde{c}_i = 3\mu_i f[x_i, x_{i+1}] + 3\lambda_i f[x_{i-1}, x_i]$, $i = 1,2,\dots,N-1$;
then $m_i \ge 0$, $i = 0,1,\dots,N$.

To prove this lemma, we simply apply Lemma 1 to the system [1,2]

$$\left.\begin{aligned} 2m_0 + \mu_0 m_1 &= \tilde{c}_0 , \\ \lambda_i m_{i-1} + 2m_i + \mu_i m_{i+1} &= \tilde{c}_i , \ i = 1,\dots,N-1 , \\ \lambda_N m_{N-1} + 2m_N &= \tilde{c}_N . \end{aligned}\right\} \tag{13}$$

From the fact that $m_i \ge 0$, $i = 0,\dots,N$ it does not follow that $S'(x) \ge 0$ for all $x \in [a,b]$. To obtain this assertion, we need stronger assumptions than in Lemma 2. Let $z_+ = z$, if $z \ge 0$; $z_+ = 0$, if $z < 0$.

Theorem 2. Let the cubic spline $S(x)$ with boundary conditions (12) interpolate the monotone data $\{f_i\}$. If

$$|\mu_0| \le 2 , \quad (\mu_0)_+ \tilde{c}_1 \le 2\tilde{c}_0 \le 12 f[x_0, x_1] , \tag{14}$$

$$|\lambda_N| \le 2 , \quad (\lambda_N)_+ \tilde{c}_{N-1} \le 2\tilde{c}_N \le 12 f[x_{N-1}, x_N] , \tag{15}$$

$$\lambda_i f[x_{i-1}, x_i] \le (1 + \lambda_i) f[x_i, x_{i+1}] , \quad i = 1,\dots,N-1 , \tag{16}$$

$$\mu_i f[x_i, x_{i+1}] \le (1 + \mu_i) f[x_{i-1}, x_i] , \quad i = 1,\dots,N-1 , \tag{17}$$

then $S'(x) \ge 0$, $x \in [a,b]$.

Proof. For $x \in [x_i, x_{i+1}]$ we have [2]

$$S'(x) = \varphi(t, m_i, m_{i+1}) = 6t(1-t) f[x_i, x_{i+1}] + (1 - 4t + 3t^2) m_i - (2t - 3t^2) m_{i+1}$$

It is easy to chack that from (14)-(17) the hypotheses of Lemma 2 follow. Thus, $m_i \ge 0$, $i = 0,\dots,N$. Now from (13) we conclude that

$m_i \leq \tilde{C}_i/2 = 3(\mu_i f[x_i, x_{i+1}] + \lambda_i f[x_{i-1}, x_i])/2$, $i = 1, \dots, N-1$.

Taking (16), (17) into account we obtain

$$0 \leq m_i \leq 3f[x_i, x_{i+1}] \ , \quad 0 \leq m_i \leq 3f[x_{i-1}, x_i], \ i = 1, \dots, N-1 . \qquad (18)$$

The function $\varphi(t, m_i, m_{i+1})$ is linear on variables m_i, m_{i+1}. Therefore, to establish the inequality $S'(x) \geq 0$, $x \in [x_i, x_{i+1}]$, $i = 1, \dots, N-2$, it is sufficient, in view of (18), to check on $t \in [0,1]$ the conditions: $\varphi(t,0,0) \geq 0$, $\varphi(t,0,\delta_i) \geq 0$, $\varphi(t,\delta_i,0) \geq 0$, $\varphi(t,\delta_i,\delta_i) \geq 0$, where $\delta_i = 3f[x_i, x_{i+1}]$. It is easy to see that they are fulfilled. In the same way we obtain the assertion of the theorem for the intervals $[x_0, x_1], [x_{N-1}, x_N]$, taking into account that

$$m_0 = (\tilde{C}_0 - \mu_0 m_1)/2 , \quad m_N = (\tilde{C}_N - \lambda_N m_{N-1})/2 .$$

Remark. The assumptions $|\mu_0| \leq 2$, $|\lambda_N| \leq 2$ are only used for intervals $[x_0, x_1]$, $[x_{N-1}, x_N]$. To obtain the assertion of Theorem 2 on $x \in [x_1, x_{N-1}]$, it is sufficient to suppose $\mu_0 < 4$, $\lambda_N < 4$.

5. Convex and monotone parabolic splines.

By the same technique used to develop sufficient conditions of monotonicity and convexity for cubic splines similar conditions can be obtained in a case of parabolic spline. We give the corresponding results without proofs.

Let $\bar{x}_i = (x_i + x_{i+1})/2$, $i = 0, \dots, N-1$. Following [16] we define the interpolating parabolic spline $S_2(x) \in C^1[a,b]$ as a function which agrees with some polynomial of degree 2 on each interval $[x_0, \bar{x}_1]$, $[\bar{x}_i, \bar{x}_{i+1}]$, $i = 0, \dots, N-1$; $[\bar{x}_{N-1}, x_N]$ and interpolate data f_i in knots of the mesh Δ. We denote $m_i = S_2'(x_i)$, $M_i = S_2''(x_i)$, $i = 0, \dots, N$.

Theorem 3. Let the parabolic spline $S_2(x)$ satisfy boundary conditions $3M_0 + \lambda_0 M_1 = d_0$, $\mu_N M_{N-1} + 3M_N = d_N$, where $\lambda_0 < 9$, $\mu_N < 9$, $d_0 \geq 0$, $d_N \geq 0$, and interpolate convex data $\{f_i\}$. If

$$3d_i - \lambda_i d_{i+1} - \mu_i d_{i-1} \geq 0 , \ i = 0, \dots, N,$$

where $d_i = 8f[x_{i-1}, x_i, x_{i+1}]$, $i = 1, 2, \dots, N-1$; $d_{-1} = d_{N+1} = 0$, then

$$S''(x) \geq 0 , \ x \in [a, b].$$

Remark. Convexity conditions for a parabolic spline are weaker than analogous conditions for a cubic spline (Theorem 1).

Theorem 4. Let the parabolic spline $S_2(x)$ satisfy boundary conditions $3m_0 + \mu_0 m_1 = \tilde{C}_0$, $\lambda_N m_{N-1} + 3m_N = \tilde{C}_N$ and interpolate monotone data $\{f_i\}$. If

$$|\mu_0| \leq 3 \ , \quad (\mu_0)_+ \tilde{C}_1 \leq 3\tilde{C}_0 \leq 6(3 + \mu_0) f[x_0, x_1],$$
$$|\lambda_N| \leq 3 \ , \quad (\lambda_N)_+ \tilde{C}_{N-1} \leq 3\tilde{C}_N \leq 6(3 + \lambda_N) f[x_{N-1}, x_N],$$

$$2\lambda_i f[x_{i-1},x_i] \le (1+2\lambda_i) f[x_i,x_{i+1}] \;,\; i=1,2,\dots,N-1,$$
$$2\mu_i f[x_i,x_{i+1}] \le (1+2\mu_i) f[x_{i-1},x_i] \;,\; i=1,2,\dots,N-1,$$

where $\tilde{c}_j = 4(\lambda_j f[x_{j-1},x_j] + \mu_j f[x_j,x_{j+1}])$, $j=1,N-1$, then

$$S_2'(x) \ge 0 \,,\quad x\in[a,b] \,.$$

6. Rational splines. We consider the rational splines supposed in [11]. In that paper the possibility of their application to interpolation which preserves monotonicity and convexity was established. Some interesting, from practical point of view, examples of rational splines are given in [10].

We define interpolating rational spline $S_R(x)\in C^2[a,b]$ as a function which on each $[x_i,x_{i+1}]$ has the form

$$S_R(x) = (1-t)f_i + t f_{i+1} + \tag{19}$$
$$+ C_i\left[\frac{t^3}{1+\varphi_i(t)} - t\right] + D_i\left[\frac{(1-t)^3}{1+\psi_i(t)} - (1-t)\right],$$

where C_i, D_i are numerical coefficients, $t=(x-x_i)/h_i$, $\varphi_i(t), \psi_i(t)\in C^2[0,1]$ are given functions. We suppose that

$$\begin{aligned}&\varphi_i(t) > -1 \,,\; \psi_i(t) > -1 \,,\; t\in[0,1]\,;\; \varphi_i(1)=\psi_i(0)=0,\\ &\varphi_i'(1)\le 0 \,,\; \psi_i'(0)\ge 0 \,,\; i=0,1,\dots,N-1 \,.\end{aligned} \tag{20}$$

Clearly, $S_R(x_i)=f_i$, $i=0,\dots,N$. The coefficients C_i, D_i are determined by the condition $S_R(x)\in C^2[a,b]$. As it follows from [11] there are a lot of functions φ_i, ψ_i such that the interpolating rational spline exists and is unique. In particular, supposing $\varphi_i(t) = \psi_i(t)\equiv 0$, $i=0,1,\dots,N-1$, we obtain the classical C^2 cubic spline.

Let

$$m_i = S_R'(x_i) \,,\quad M_i = S_R''(x_i) \,,\quad i=0,\dots,N;$$
$$\Delta_i = 1 - [2+\psi_i'(0)][2-\varphi_i'(1)] \,,$$
$$u_i = \{6 - 6\varphi_i'(1) - \varphi_i''(1) + 2[\varphi_i'(1)]^2\}^{-1} \,,\; P_i = [u_i\Delta_i]^{-1},$$
$$v_i = \{6 + 6\psi_i'(0) - \psi_i''(0) + 2[\psi_i'(0)]^2\}^{-1} \,,\; Q_i = [v_i\Delta_i]^{-1},$$
$$i=0,\dots,N-1 \,.$$

Then we have

$$\left.\begin{aligned}C_i &= \{(f_{i+1}-f_i)[3+\psi_i'(0)] - h_i m_i - h_i m_{i+1}[2+\psi_i'(0)]\}/\Delta_i \,,\\ D_i &= \{(f_{i+1}-f_i)[\varphi_i'(1)-3] + h_i m_i[2-\varphi_i'(1)] + h_i m_{i+1}\}/\Delta_i \,,\end{aligned}\right\} \tag{21}$$

$$C_i = h_i^2 u_i M_{i+1} \,,\quad D_i = h_i^2 v_i M_i \,,\quad i=0,1,\dots,N-1 \,. \tag{22}$$

In [11] it was proved that m_i satisfy the equations

$$-\lambda_i P_{i-1} m_{i-1} - L_i m_i - \mu_i Q_i m_{i+1} = -g_i\ ,\quad i=1,\dots,N-1\ , \tag{23}$$

where $L_i = \lambda_i P_{i-1}[2+\psi'_{i-1}(0)] + \mu_i Q_i[2-\varphi'_i(1)]$,

$$g_i = \lambda_i P_{i-1}[3+\psi'_{i-1}(0)] f[x_{i-1},x_i] + \mu_i Q_i[3-\varphi'_i(1)] f[x_i,x_{i+1}].$$

To these must be adjoined the equations that follow from boundary conditions for the spline. Below we consider only two types of boundary conditions:

$$S'_R(x_K) = f'_K\ ,\ (m_K = f'_K)\ ,\quad K=0,N; \tag{24}$$

$$S''_R(x_K) = f''_K\ ,\ (M_K = f''_K)\ ,\quad K=0,N. \tag{25}$$

If (20) is satisfied and

$$P_i < 0\ ,\ Q_i < 0\ ,\quad i=0,1,\dots,N-1, \tag{26}$$

then the rational interpolating spline $S_R(x)$ with boundary conditions (24) or (25) exists and is unique [11].

We also need the equations for the unknowns M_i. As (23), they follow from the assumption $S_R(x) \in C^2[a,b]$:

$$\mu_i u_{i-1} M_{i-1} + w_i M_i + \lambda_i v_i M_{i+1} = f[x_{i-1},x_i,x_{i+1}],\ i=1,\dots,N-1, \tag{27}$$

where $w_i = \mu_i v_{i-1}[2-\varphi'_{i-1}(1)] + \lambda_i u_i[2+\psi'_i(0)]$.

On the concrete example of the functions

$$\varphi_i(t) = p_i(1-t^\alpha)\ ,\quad \psi_i(t) = q_i[1-(1-t)^\alpha]\ , \tag{28}$$

where $\alpha > 0,\ p_i \geq 0,\ q_i \geq 0$ (for $\alpha = 1$ we have a spline from [2,12]) in [11] it is shown that by the choice of the parameters p_i, q_i we can construct monotone and convex interpolation. Namely, let there for some K be $f[x_{i-1},x_i,x_{i+1}] > 0$, $i=K,K+1$. Then for all sufficiently large $q_{K-1}, q_K, p_K, p_{K+1}$ there will be $S''_R(x) \geq 0, x \in [x_K, x_{K+1}]$. If $f_{K-1} < f_K < f_{K+1} < f_{K+2}$, then for sufficiently large $q_{K-1}, q_K, p_K, p_{K+1}$ we have $S'_R(x) \geq 0$, $x \in [x_K, x_{K+1}]$. It is proved, too, that

$$\lim_{p_K, q_K \to \infty} S_R(x) = (1-t)f_K + t f_{K+1}\ ,\quad x \in [x_K, x_{K+1}]. \tag{29}$$

Other types of functions φ_i, ψ_i with similar properties are given in [11]. Some interesting constructions of splines are obtained if

$$\varphi_i(t) = t^3/\xi_i(t) - 1\ ,\quad \psi_i(t) = (1-t)^3/\eta_i(t) - 1, \tag{30}$$

where $\xi_i(t), \eta_i(t) \in C^2[0,1]$ and their properties must be in accordance with (20), (26). In this case formula (19) of spline $S_R(x)$ takes a simple "non-rational" form

$$S_R(x) = (1-t)f_i + t f_{i+1} + C_i[\xi_i(t) - t] + D_i[\eta_i(t) - (1-t)]\ . \tag{31}$$

We note the following special kind of functions ξ_i, η_i:

$$\xi_i(t) = (t-\alpha_i)_+^3/(1-\alpha_i)^3, \quad \eta_i(t) = (\beta_i - t)_+^3/\beta_i^3, \tag{32}$$

where $0 \le \alpha_i < 1, 0 < \beta_i \le 1$. In this case the spline $S_R(x)$ is C^2 cubic spline with additional knots in points $x_i + \alpha_i h_i, x_i + \beta_i h_i$ (if $\alpha_i = \beta_i$ for all i we obtain the spline from [9]). The parameters α_i, β_i play the same role as p_i, q_i in (28). For example, if $f_{K-1} < f_K < f_{K+1} < f_{K+2}$ and α_i, α_{i+1} are close to 1; β_{i-1}, β_i are close to 0, then $S'(x) \ge 0$ on $[x_K, x_{K+1}]$. Note that for $\beta_i < \alpha_i$ the spline $S_R(x)$ on intervals $[x_i + \beta_i h_i], [x_i + \alpha_i h_i]$ is a polynomial of degree one.

7. Convex interpolation by rational splines.

Theorem 5. Let the rational spline $S_R(x)$ given by (19) interpolate convex data $\{f_i\}$ and satisfy boundary conditions (25), where $f_0'' \ge 0$, $f_N'' \ge 0$. If

$$\begin{gathered} \varphi_i(t) = \varphi_i(1-t), \ \varphi_i(t) > -1, \ \Phi_i''(t) \ge 0, \ t \in [0,1], \\ \varphi_i(1) = 0, \ \varphi_i'(1) \le 0, \quad i = 0, \ldots, N-1; \end{gathered} \tag{33}$$

$$2 - \varphi_i'(1) \ge \max\{d_i/(\lambda_i d_{i+1}), d_{i+1}/(\mu_i d_i)\}, i = 1, \ldots, N-2; \tag{34}$$

$$6 - 6\varphi_0'(1) - \varphi_0''(1) + 2[\varphi_0'(1)]^2 \ge f_0''/d_1, \tag{35}$$

$$6 - 6\varphi_{N-1}'(1) - \varphi_{N-1}''(1) + 2[\varphi_{N-1}'(1)]^2 \ge f_N''/d_{N-1}, \tag{36}$$

where $\Phi_i(t) = t^3/[1+\varphi_i(t)]$, $d_i = f[x_{i-1}, x_i, x_{i+1}]$, then

$$S_R''(x) \ge 0, \quad x \in [a,b].$$

Proof. From (33) it follows $\Delta_i < 0, P_i = Q_i < 0, u_i = v_i = [\Phi_i''(1)]^{-1} > 0$. Thus, $S_R(x)$ exists and is unique. From (19), (22) we have

$$S_R''(x) = u_i \Phi_i''(t) M_{i+1} + v_i \Phi_i''(1-t) M_i, \ x \in [x_i, x_{i+1}].$$

Therefore, to prove the theorem, it is sufficient to show that $M_i \ge 0$, $i = 0, \ldots, N$. Applying Lemma 1 to the system of equations (25), (27) we obtain that $M_i \ge 0, i = 0, \ldots, N$, if

$$\lambda_i\{d_i - u_i d_{i+1}/w_{i+1}\} + \mu_i\{d_i - u_{i-1} d_{i-1}/w_{i-1}\} \ge 0, \ i = 1, \ldots, N-1, \tag{37}$$

where $w_0 = w_N = 1$, $d_0 = f_0''$, $d_N = f_N''$. Taking into account the inequalities

$$\begin{gathered} u_i/w_{i+1} \le \{\mu_{i+1}[2 - \varphi_i'(1)]\}^{-1}, \ i = 1, \ldots, N-2, \\ u_{i-1}/w_{i-1} \le \{\lambda_{i-1}[2 - \varphi_{i-1}'(1)]\}^{-1}, \ i = 2, \ldots, N-1. \end{gathered}$$

and conditions (34)-(36), we obtain (37) which completes the proof of the theorem.

We give two examples to illustrate conditions (34)-(36).

Example 1. Let $\varphi_i(t) = p_i(1-t)$, $i = 0,\dots,N-1$ (this is (28) for $\alpha = 1$). Then $S_R(x)$ is convex, if

$$2 + p_i \geq \max\{d_i/(\lambda_i d_{i+1}),\, d_{i+1}/(\mu_i d_i)\},\ i = 1,\dots,N-2$$

$$6 + 6p_0 + 2p_0^2 \geq f_0''/d_1,\quad 6 + 6p_{N-1} + 2p_{N-1}^2 \geq f_N''/d_{N-1}.$$

In view of (29) the spline $S_R(x)$ is in fact linear interpolation, if p_i are large. Therefore, from all p_i that guarantee the convexity of $S_R(x)$, the minimal p_i must be taken.

Example 2. Let $\varphi_i(t) = t^3/\xi_i(t) - 1$, $\xi_i(t) = (t-\alpha_i)_+^3/(1-\alpha_i)^3$, (see (30), (32)). The parameters α_i that give the arrangement of additional knots $x_i + \alpha_i h_i$, $x_{i+1} - \alpha_i h_i$ are determined from (34)-(36), where

$$\varphi_i'(1) = -3\alpha_i/(1-\alpha_i),\quad \varphi_i''(1) = 6\alpha_i(1+\alpha_i)/(1-\alpha_i)^2.$$

8. Monotone interpolation by rational splines.

Theorem 6. Let the rational spline $S_R(x)$ given by (19) interpolate monotone data $\{f_i\}$ and satisfy boundary conditions (24) with $f_0' \geq 0$, $f_N' \geq 0$. If (20), (26) hold and

$$2 + \tfrac{2}{3}\varphi_i'(0) \geq \frac{f[x_{i-1},x_i]}{f[x_i,x_{i+1}]},\quad 2 - \tfrac{2}{3}\varphi_i'(1) \geq \frac{f[x_{i+1},x_{i+2}]}{f[x_i,x_{i+1}]}, \tag{38}$$

$$\begin{gathered} 3t[1+\varphi_i(t)] - t^2\varphi_i'(t) \leq [3-\varphi_i'(1)][1+\varphi_i(t)]^2, \\ 3(1-t)[1+\varphi_i(t)] + (1-t)^2\varphi_i'(t) \leq [3+\varphi_i'(0)][1+\varphi_i(t)]^2, \\ i = 0,\dots,N-1;\quad f[x_{-1},x_0] = f_0',\ f[x_N,x_{N+1}] = f_N', \end{gathered} \tag{39}$$

then $S_R'(x) \geq 0$, $x \in [a,b]$.

The proof is analogous to that of Theorem 2. First, taking into account (38) and applying Lemma 1 to equations (23), (24), we have $m_i \geq 0$, $i = 0,\dots,N$. To complete the proof, we use (39) and the fact that $S_R'(x)$ on $[x_i,x_{i+1}]$ is linear in arguments m_i, m_{i+1}.

9. Conclusion. By the application of the rational splines we can construct convex or monotone interpolating spline for any convex or monotone data. In practical use of the obtained results we recommend

first to check the conditions of monotonicity (convexity) for cubic spline (Theorems 1 and 2). Just only on the intervals that are neighbouring to the points, in which these conditions are violated, it is necessary to use the rational spline. This procedure may be realized very easily since the cubic spline is a special case of rational splines.

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