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ON SOME INEQUALITIES IN SPACES OF INTEGRABLE FUNCTIONS

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In [4] we obtained a Bernstein type inequality for fractional order derivatives in an Orlicz space $L^\varphi(-\infty,\infty)$ with φ-function depending on a parameter, applying a theorem on continuity of the translation operator in this space. Here, we extend the last theorem to a larger class of spaces of measurable functions in $(-\infty,\infty)$, which makes it possible to obtain the Bernstein type inequality in this class.

1. Let $\varphi(x,u)$ be a φ-function depending on the parameter $x\in R=(-\infty,\infty)$, s-convex with respect to $u\geq 0$ for a.e. $x\in R$ with some $s\in(0,1\rangle$ (see [2]). Let S_+ be the set of all finite a.e., measurable functions ψ on R with non-negative values including ∞, with equality a.e. Let Φ be a functional defined in S_+ such that for every $\psi,\psi_1,\psi_2\in S_+$

(a) $0\leq\Phi(\psi)\leq\infty$, $\Phi(0)=0$,

(b) $\Phi(\psi_1+\psi_2)\leq\Phi(\psi_1)+\Phi(\psi_2)$,

(c) if $\psi_1\leq\psi_2$, then $\Phi(\psi_1)\leq\Phi(\psi_2)$,

(d) $\Phi(a\psi)=a\,\Phi(\psi)$ for $a>0$,

(e) $\Phi(\varphi(\cdot,|f(\cdot+h)|))\leq\Phi(\varphi(\cdot-h,|f(\cdot)|))$ for all measurable functions f on R and all real h.

Then $\varrho(f)=\Phi(\varphi(\cdot,|f(\cdot)|))$ is an s-convex pseudomodular in the space S of all measurable, finite a.e. functions f with equality a.e. We shall be concerned with the modular space $X_\varrho=\{f\in S\colon \varrho(af)\to 0$ as $a\to 0+\}$ with the s-homogeneous pseudonorm

$$\|f\| = \inf \{u > 0: \varrho(u^{-1/s} f) \le 1\}.$$

We shall say that the function φ satisfies the condition (b_h) with some $h \in R$, if there exist a set $A \subset R$ of Lebesgue measure 0, a constant $c > 0$ and a non-negative, measurable function $F(\cdot, h)$ on R, satisfying the condition $S_h = \Phi(F(\cdot, h)) < \infty$ such that for every $u \geqslant 0$ and $x \in R \setminus A$ there holds the inequality

$$\varphi(x-h, u) \le \varphi(x, cu) + F(x, h).$$

If φ satisfies (b_h) for every $h \in R$ with the same constant c, set A and function F, and if $S_\infty = \sup_h S_h < \infty$, then we shall say that φ satisfies the condition (B_∞) (see [4]). Let us remark that if φ does not depend on x, then it satisfies (B_∞) with $c = 1$ and $F(x, h) = 0$.

The following theorem on continuity of the translation operator holds in X_ϱ:

Theorem 1. If the function φ satisfies the condition (b_h) with some $h \in R$, then for every $f \in X_\varrho$,

$$\|f(\cdot + h)\| \le \begin{cases} 2c^s \max(1, S_h)\|f\| & \text{if } S_h \geqslant \frac{1}{2} \\ \frac{c^s}{1-S_h}\|f\| & \text{if } 0 \le S_h < \frac{1}{2}. \end{cases}$$

Proof. Let an $\varepsilon > 0$ and a $d \geqslant 1$ be given and let $f \in X_\varrho$. Then

$$\varrho(\varepsilon^{-1} d^{-1/s} f(\cdot + h)) \le \Phi(\varphi(\cdot - h, \varepsilon^{-1} d^{-1/s} |f(\cdot)|)) \le d^{-1} \Phi(\varphi(\cdot - h, \varepsilon^{-1} |f(\cdot)|)) \le$$
$$\le d^{-1} \Phi(\varphi(\cdot, c\,\varepsilon^{-1} |f(\cdot)|) + F(\cdot, h)) \le d^{-1} \Phi(\varphi(\cdot, c\,\varepsilon^{-1} |f(\cdot)|)) + d^{-1} \Phi(F(\cdot, h)) =$$
$$d^{-1} \varrho(\varepsilon^{-1} cf) + d^{-1} S_h.$$

Hence

$$(1) \quad \varrho(2^{-1/s} d^{-1/s} \varepsilon^{-1} f(\cdot + h)) \le \frac{1}{2} \varrho(d^{-1/s} \varepsilon^{-1} f(\cdot + h)) \le \frac{1}{2d} \varrho\left(\frac{cf}{\varepsilon}\right) + \frac{1}{2d} S_h.$$

Now, let $d = \max(1, S_h)$. Then

$$\varrho(2^{-1/s} d^{-1/s} \varepsilon^{-1} f(\cdot + h)) \le \frac{1}{2} \varrho\left(\frac{cf}{\varepsilon}\right) + \frac{1}{2}.$$

Let us suppose $\|f\| < (\varepsilon/c)^s$, i.e. $\|cf/\varepsilon\| < 1$. Then $\varrho(2^{-1/s} d^{-1/s} \varepsilon^{-1} f(\cdot + h)) < 1$, whence $\|f(\cdot + h)\| \le 2d\,\varepsilon^s$. Consequently, $\|f(\cdot + h)\| \le 2dc^s \|f\|$.

Suppose now that $S_h < 1$ and apply (1) with $d = 1$. Then $\varrho(\varepsilon^{-1} f(\cdot + h)) \le \varrho(\varepsilon^{-1} cf) + S_h$. Thus, if $\varrho(\varepsilon^{-1} cf) \leqslant 1 - S_h$, then $\varrho(\varepsilon^{-1} f(\cdot + h)) \le 1$. But

$(1-S_h)^{-1}\varrho(\varepsilon^{-1}cf) \le \varrho\left(\varepsilon^{-1}(1-S_h)^{-1/s}cf\right)$. Hence, supposing $\|f\| < c^{-s}\varepsilon^s(1-S_h)$, we obtain $\varrho(\varepsilon^{-1}cf) \le 1-S_h$. Consequently, $\varrho(\varepsilon^{-1}f(\cdot+h)) \le 1$, whence $\|f(\cdot+h)\| \le \varepsilon^s$. Thus, $\|f(\cdot+h)\| \le (1-S_h)^{-1}c^s\|f\|$.

Let us still remark that for $S_h = \frac{1}{2}$, both constants

$$C_1 = 2c^s \max(1, S_h) \text{ and } C_2 = (1-S_h)^{-1}c^s$$

are equal. For $S_h > \frac{1}{2}$ we have $C_1 < C_2$, and for $S_h < \frac{1}{2}$, there is $C_2 < C_1$.

2. We shall give some examples of functionals Φ and respective modular spaces X_ϱ.

2.1. Defining $\Phi(\psi) = \int_{-\infty}^{\infty}\psi(t)\,dt$, Φ satisfies all the required assumptions (a) -(e) and $\varrho(f) = \int_{-\infty}^{\infty}\varphi(x, |f(x)|)\,dx$, $X_\varrho = L^\varphi(-\infty, \infty)$. Theorem 1 is then given in [4].

2.2. Taking $\Phi(\psi) = \int_{-\pi}^{\pi}\psi(t)\,dt$, $\varphi(x,u)$ 2π-periodic with respect to x, Φ satisfies (a) -(e) again and $\varrho(f) = \int_{-\pi}^{\pi}\varphi(x, |f(x)|)\,dx$, $X_\varrho = L^\varphi_{2\pi}$, where $S = S_{2\pi}$ is the space of 2π-periodic, measurable, finite a.e. functions.

2.3. Let $\Phi(\psi) = \frac{1}{l}\sup_x \int_x^{x+l}\psi(t)\,dt$ with a fixed $l > 0$. Then Φ satisfies (a) - (e), $\varrho(f) = \varrho^\varphi_l(f) = \frac{1}{l}\sup_x \int_x^{x+l}\varphi(t, |f(t)|)\,dt$, and $X_\varrho = S^\varphi_l$ is a generalized Stepanov space ; in the case of φ independent of the parameter, S^φ_l was defined in [3]. It is easily seen that for every $f \in S$, there exists the limit $\lim_{l\to\infty}\varrho^\varphi_l(f)$, may be infinite ; this is proved in [3] in case of φ independent of the parameter, and the proof in general case follows the same lines.

2.4. Let $\Phi(\psi) = \lim_{l\to\infty}\sup_x \frac{1}{l}\int_x^{x+l}\psi(t)\,dt$, then Φ satisfies (a) - (e), $\varrho(f) = \varrho^\varphi_W(f) = \lim_{l\to\infty}\sup_x \frac{1}{l}\int_x^{x+l}\varphi(x, |f(x)|)\,dx$, and $X_\varrho = W^\varphi$ is a generalized Weyl space ; in the case of φ independent of the parameter, W was defined also in [3].

Theorem 2. Let h_ν be a sequence of real numbers, $\nu = 0, \pm1, \pm2, \ldots$ and let $f \in S$. Let us suppose that the series $g(x) = \sum_{\nu=-\infty}^{\infty} c_\nu f(x+h_\nu)$ is convergent for all real x, and that the φ-function φ satisfies the condition (B_∞). Then in each of the spaces X_ϱ from 2.1 - 2.4 there holds the inequality $\|g\| \le C\sum_{\nu=-\infty}^{\infty}|c_\nu|^s\|f\|$, where

(2) $$C = \begin{cases} 2c^s \max(1, S_\infty) & \text{for } S_\infty \geq \frac{1}{2} \\ \dfrac{c^s}{1 - S_\infty} & \text{for } 0 \leq S_\infty < \frac{1}{2}. \end{cases}$$

Proof. Let $g_n(x) = \sum_{\nu=-n}^{n} c_\nu f(x+h)$, then $g(x) = \lim_{n\to\infty} g_n(x)$ for all x. Moreover, in each of the spaces X_ϱ under consideration we have, by Theorem 1,

$$\| g_n \| \leq \sum_{\nu=-n}^{n} |c_\nu|^s \|f(\cdot+h)\| \leq C \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \|f\|.$$

Now, let us consider the case 2.1, then Fatou lemma gives

$$\varrho(u^{-1/s} g) \leq \underline{\lim}_{n\to\infty} \int_{-\infty}^{\infty} \varphi(x, u^{-1/s}|g_n(x)|)dx = \underline{\lim}_{n\to\infty} \varrho(u^{-1/s} g_n).$$

Supposing $u > \sup_n \| g_n\|$, we obtain $\varrho(u^{-1/s} g_n) \leq 1$. Hence $\varrho(u^{-1/s} g) \leq 1$, and so $\|g\| \leq u$. Consequently, $\| g \| \leq \sup_n \| g_n \| \leq C \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \| f\|$.

The result in case 2.2 is obtained in the same way.

Considering the case 2.3 we observe that application of Fatou lemma gives

$$\frac{1}{l} \int_x^{x+l} \varphi(t, u^{-1/s} | g(t)|)dt \leq \underline{\lim}_{n\to\infty} \varrho_l^\varphi(u^{-1/s} g_n)$$

for $u > 0$, and hence again $\varrho_l^\varphi(u^{-1/s} g) \leq \underline{\lim}_{n\to\infty} \varrho_l^\varphi(u^{-1/s} g_n)$. The proof is now concluded as in case 2.1.

Coming over to the case 2.4 let us first remark that denoting by $\| \ \|_W^\varphi$ the norm in W^φ and by $\| \ \|_l^\varphi$ - the norm in S_l^φ, we have $\|f\|_W^\varphi = \lim_{l\to\infty} \|f\|_l^\varphi$ for every $f \in W^\varphi$. In case of φ independent of the parameter this is proved in [3], and the proof in the general case follows the same lines. Now, the required inequality in W^φ follows, taking $l \to \infty$ on both sides of the inequality $\| g\|_l^\varphi \leq C \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \| f\|_l^\varphi$.

3. As an application of Theorem 2, we may conclude a theorem concerning a Bernstein type inequality in the above spaces. Let $\omega(t)$ be a function of bounded variation in the interval $\langle -R, R \rangle$, where $R > 0$ is fixed, and let μt and $\gamma(t)$ be defined and continuous in $\langle -R, R\rangle$. Let f be a function of the form

(3) $$f(x) = \int_{-R}^{R} \gamma(t) e^{ixt} \, d\omega(t)$$

and let

$$g(x) = \int_{-R}^{R} \mu(t) \gamma(t) e^{ixt} \, d\omega(t).$$

In the case when $\gamma(t)=1$, $\mu(t)=i^{\alpha}t^{\alpha}$ with $\alpha\geqslant 0$, $f^{(\alpha)}=g$ is the derivative of order α of f. Supposing $\mu(t)e^{iat}=\sum_{\nu=-\infty}^{\infty} c_\nu e^{ip(\nu)t}$ for $t\in\langle -R,R\rangle$, where $\sum_{\nu=-\infty}^{\infty}|c_\nu|^s<\infty$ and $p(\nu)=R^{-1}\nu\pi$ for $\nu=0,\pm1,\pm2,\ldots$ for some $a\geqslant 0$, $0<s\leq 1$, we have $g(x)=\sum_{\nu=-\infty}^{\infty} c_\nu f[p(\nu)-a+x]$ for all real x (see [1]). Thus, applying Theorem 2 with $h_\nu=p(\nu)-a$, we obtain the following result for the derivatives $f^{(\alpha)}$ of f of order α:

Corollary 1. Let X_ϱ be any of the spaces 2.1 - 2.4, where φ is a convex function depending on a parameter, satisfying the condition (B_∞). Let $f\in X_\varrho$ be of the form (3) with $\gamma(t)=1$. Then

$$\|f^{(\alpha)}\|\leq C^{\alpha}R^{\alpha}\|f\| \quad \text{for } \alpha=0,1,2,\ldots,$$

$$\|f^{(\alpha+q)}\|\leq 1.8\, C^{q+1}R^{\alpha+q}\|f\| \quad \text{if } 1<\alpha<2,\ q=0,1,2,\ldots,$$

$$\|f^{(\alpha)}\|\leq \frac{7}{\alpha}CR^{\alpha}\|f\| \quad \text{for } 0<\alpha<1,$$

where C is given by (2).

Corollary 2. Let X_ϱ be any of the spaces 2.1 - 2.4 , where is an s-convex function depending on a parameter, $s\in\left(\frac{1}{2},1\right)$, satisfying the condition (B_∞). Let $f\in X_\varrho$ be of the form (3) with $\gamma(t)=1$. Then

$$\|f^{(\alpha+q)}\|\leqslant C(\alpha,q)\,A(s,\alpha)\left\{A(s,1)\right\}^{q}R^{(\alpha+q)s}\|f\|,$$

where

$$A(s,\alpha)=2^{s+1}\alpha^{s}\pi^{-2s}\sum_{\nu=1}^{\infty}\nu^{-2s}\;2(\alpha+1)^{-s}$$

and $C(\alpha,q)$ is a positive constant.

Corollaries 1 and 2 in case of 2.1 are given in [4].

References

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