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THE GIBBS PHENOMENON IN GENERALIZED PADÉ APPROXIMATION

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1.Introduction.

If one approximate a discontinuous function by polynomials /from the Fourier series / it leads to an unusual property - the Gibbs phenomenon. The polynomials do not converge to the function near the discontinuity. The maximal value of the error is called Gibbs constant. For example , it is well known that when we approximate the function $sgn(x)$ in $(-1,+1)$ by Fourier series the Gibbs constant is

$$G = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt - 1 = 0,1789797\ldots .$$

Another important property of the approximation is the steepness. We call the value of the derivative of the approximant at the discontinuity the steepness. For the function $sgn(x)$ the steepness is $\frac{4}{\pi}(n+1)$, for an n-term Fourier series approximation. It is desirable to obtain an approximation for which the Gibbs constant is as small as possible and the steepness is as high as possible.

In this paper we consider some rational functions and we show that the generalized Padé approximants have Gibbs constants smaller than G and their steepness is higher than $C.n$.

The paper is arranged as follows. In Section 2 we consider the generalized Padé approximation in the sense of Cheney [1] . We provide proofs of the results in Section 3 .

2. Approximants for $sgn(x)$.

Here we apply a series representation for the function $sgn(x)$ in the form

$$sgn(x) = \left\{ \begin{array}{ll} -1 , & -1 \leq x < 0 \\ +1 , & 0 < x \leq 1 \end{array} \right\} = \sum_{n=0}^{\infty} d_n C_{2n+1}^{\sigma}(x) \qquad /1/$$

where $C_{2n+1}^{\sigma}(x)$ is the Gegenbauer polynomial and $d_n = \frac{\Gamma(\sigma)(2n+\sigma+1)}{\Gamma(n+\sigma+\frac{3}{2})\Gamma(\frac{1}{2}-n)}$.

The rationals

$$R_{n,m}(x) = \frac{\sum_{i=0}^{m} p_i C_{2i+1}^{\sigma}(x)}{\sum_{i=0}^{n} q_i C_{2i}^{\sigma}(x)}, \qquad /2/$$

which satisfy the relation

$$\left\{ \sum_{i=0}^{n} q_i C_{2i}^{\sigma}(x) \right\} sgn(x) - \sum_{i=0}^{m} p_i C_{2i+1}^{\sigma}(x) = O\left(C_{2n+2m+3}^{\sigma}(x) \right) \qquad /3/$$

are called the generalized Padé approximants [1]. The O-term in /3/ means a function for which the series in $C_i^{\sigma}(x)$ begins with the term $C_{2n+2m+3}^{\sigma}(x)$.

Next we shall list our main results. The solution of problem / 3 / in explicit form is

$$R_{n,m}(x) = A_{n,m} \cdot x \cdot \frac{{}_3F_2(-m, -n+\frac{1}{2}, n+m+\sigma+2; \frac{3}{2}, \frac{3}{2}; x^2)}{{}_3F_2(-n, -m-\frac{1}{2}, n+m+\sigma+\frac{3}{2}; \frac{1}{2}, 1; x^2)}, \qquad /4/$$

where the steepness $A_{n,m}$ is

$$A_{n,m} = \frac{4}{\sqrt{\pi}} \frac{n!\,\Gamma(m+\frac{3}{2})\,\Gamma(n+m+\sigma+2)}{m!\,\Gamma(n+\frac{1}{2})\,\Gamma(n+m+\sigma+\frac{3}{2})} . \qquad /5/$$

For $n=0$, $\sigma=0$ we can get the classic result. In this case the approximating polynomial is

$$R^{0}_{0,m}(x) = A^{0}_{0,m} \cdot x \cdot {}_3F_2\left(-m, m+2, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{3}{2}; x^2\right).$$

Its error function takes the highest maximum at the point $x=\frac{\alpha}{m}$, $m\to\infty$. This value is the Gibbs constant

$$G = \frac{4}{\pi}\alpha \, {}_1F_2\left(\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{3}{2}; -\alpha^2\right) - 1 .$$

Differentiating by α we get an equation for α

$${}_0F_1\left(; \tfrac{3}{2}; -\alpha^2\right) = \frac{\sin 2\alpha}{2\alpha} = 0 .$$

Its first zero is $\alpha=\frac{\pi}{2}$. The previous series considered in integral form gives the classic result

$$G = \frac{4}{\pi}\int_0^{\alpha} \frac{\sin 2u}{2u}\,du - 1 = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt - 1 .$$

The steepness is $A^{0}_{0,m} = \frac{4}{\pi}(m+1)$.

Second , we consider the case $m=0$, the reciprocal polynomial case. The approximants are

$$R_{n,0}(x) = \frac{A_{n,0}\cdot x}{{}_3F_2\left(-n, n+\sigma+\tfrac{3}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, 1, x^2\right)} .$$

Its error function takes the highest maximum at the point $x=\frac{\beta}{n}$, $n\to\infty$. By elementary calculations one can prove that β is the root of the equation

$$J_0(2\beta)=0 ,$$

where $J_0(x)$ is the Bessel function of order zero. From its first root we get

$$\beta = 1{,}2024127788\ldots \quad .$$

Therefore the Gibbs constant is

$$G_{0,1} = \frac{1}{\int_0^{2\beta} \frac{J_1(u)}{u}\,du} - 1 = 0{,}051356067\ldots \quad .$$

That is, in this case the Gibbs constant is approximately

5%. The steepness is

$$A_{n,c}=\frac{2\cdot n!\,\Gamma(n+\sigma+2)}{\Gamma(n+1/2)\Gamma(n+\sigma+3/2)}=2(n+1)a_n\ ,$$

where $a_n\approx 1$ for moderate and large values of n.

Finally, the most interesting case is $n=m$. The approximants are

$$R_{n,n}(x)=A_{n,n}\cdot x\cdot\frac{{}_3F_2\left(-n,-n+\frac{1}{2},2n+\sigma+2;\frac{3}{2},\frac{3}{2};x^2\right)}{{}_3F_2\left(-n,-n-\frac{1}{2},2n+\sigma+\frac{3}{2};\frac{1}{2},1;x^2\right)}\ ,$$

where $A_{n,n}=\dfrac{2(2n+1)\Gamma(2n+\sigma+2)}{\sqrt{\pi}\,\Gamma(2n+\sigma+3/2)}$.

The error function takes the highest maximum at the point $x=\frac{\gamma}{n^{3/2}}\ ;\ n\to\infty$. The number γ is the root of the equation

$$\sum_{k=0}^{\infty}\frac{(-2\gamma^2)^k}{(k!)^2(3/2)_k}=0,$$

and its value is $\gamma=0{,}9510208 74\ldots$. The Gibbs constant is given by the formula

$$G_{1,1}=\frac{8\gamma}{\sqrt{2\pi}}\frac{{}_0F_2\left(;\frac{3}{2},\frac{3}{2};2\gamma^2\right)}{{}_0F_2\left(;\frac{1}{2},1,2\gamma^2\right)}-1=0{,}0081489002\ldots\ .$$

Note. Semerdjiev and Nedelchev [2] performed a numerical experiment for determining $G_{1,1}$ enabling them to state that $G_{1,1}$ does not exceed 2%.

The steepness is $A_{n,n}=4\sqrt{\frac{2}{\pi}}n^{3/2}\cdot b_n$, where $b_n\approx 1$ for moderate $(n>10)$ and large values of n.

3.Proofs.

First we shall prove formula /4/. Let us consider a more generalized series expansion for $\operatorname{sgn}(x)$ like /1/

$$x^{2k}\operatorname{sgn}(x)=k!\left(\tfrac{1}{2}\right)_k\sum_{j=0}^{\infty}\frac{\Gamma(\sigma)(2j+\sigma+1)}{\Gamma(k+j+\sigma+\frac{3}{2})\Gamma(k-j+\frac{1}{2})}C^{\sigma}_{2j+1}(x),\ k=0,1,2,\ldots\ .$$

Next, multiplying it by numbers $q_k\ (k=0,1,2,\ldots,n)$, then summing these equations , we get

$$\left(\sum_{k=0}^{n}q_kx^{2k}\right)\operatorname{sgn}(x)=\frac{\Gamma(\sigma)}{\pi}\cdot\sum_{j=0}^{\infty}(2j+\sigma+1)\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\sigma+\frac{3}{2})}C^{\sigma}_{2j+1}(x)\sum_{k=0}^{n}q_k\frac{k!(\frac{1}{2})_k}{(\sigma+\frac{3}{2}+j)_k(\frac{1}{2}-j)_k}\ .$$

We want to determine the coefficients in such a manner that the following equations are satisfied

$$\sum_{k=0}^{n} q_k \frac{k!\,(1/2)_k}{(\sigma+\frac{3}{2}+j)_k(\frac{1}{2}-j)_k} = 0, \quad j = m+1, m+2, \ldots m+n .$$

In this case the numerator polynomial will be

$$\frac{\Gamma(\sigma)}{\pi} \sum_{j=0}^{m} (2j+\sigma+1) \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\sigma+3/2)} C^{\sigma}_{2j+1}(x) \sum_{k=0}^{n} q_k \frac{k!\,(1/2)_k}{(\sigma+\frac{3}{2}+j)_k(\frac{1}{2}-j)_k} .$$

To solve the previous equations let us suppose for a moment that

$$q_k = \frac{(-n)_k(-m-\frac{1}{2})_k(n+m+\sigma+3/2)_k}{(k!)^2(1/2)_k} .$$

Consider now the sum

$$S = \sum_{k=0}^{n} \frac{(-n)_k(-m-\frac{1}{2})_k(n+m+\sigma+\frac{3}{2})_k}{k!\,(1/2-j)_k(\sigma+3/2+j)_k} = {}_3F_2\left(-n, -m-\tfrac{1}{2}, n+m+\sigma+\tfrac{3}{2}; \tfrac{1}{2}-j, \sigma+\tfrac{3}{2}+j; 1\right).$$

S is a Saalschütz type hypergeometric sum and therefore it is summable by factorial functions. Really [3],

$$S = \frac{(1+m-j)_n(-1-n-m-\sigma-j)_n}{(1/2-j)_n(-\frac{1}{2}-j-n-\sigma)_n} .$$

It is not difficult to see that all products differ from zero except the first one. Further, when j runs from $m+1$ to $m+n$ then $1+m-j$ runs from 0 to $1-n$ by -1. Therefore $S=0$ for all j ($j = m+1, m+2, \cdots, m+n$). We have thus proved the form of the denominator polynomial. To get the explicit form of the numerator polynomial we apply the value of S for $j = 0, 1, \ldots, m$

$$Z = \frac{\Gamma(\sigma)}{\pi} \sum_{j=0}^{m} (2j+\sigma+1) \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\sigma+3/2)} C^{\sigma}_{2j+1}(x) \frac{(1+m-j)_n(-1-n-m-j-\sigma)_n}{(1/2-j)_n(-1/2-n-j-\sigma)_n} .$$

Taking the power form of the Gegenbauer polynomial

$$C^{\sigma}_{2j+1}(\sigma) = (-1)^j \frac{\sigma(\sigma+1)_j}{j!} 2x\, {}_2F_1(-j, j+\sigma+1; \tfrac{3}{2}; x^2) ,$$

we get

$$Z = 2x \frac{\Gamma(\sigma+1)\Gamma(n+m+1)\Gamma(n+m+\sigma+2)}{\Gamma(m+1)\Gamma(n+\frac{1}{2})\Gamma(n+\sigma+\frac{3}{2})\Gamma(m+\sigma+2)} \sum_{j=0}^{m} (2j+\sigma+1)\, {}_2F_1(-j, j+\sigma+1; \tfrac{3}{2}; x^2) \frac{(-m)_j(\sigma+1)_j(\frac{1}{2}-n)_j(n+m+\sigma+2)_j}{j!\,(-n-m)_j(n+\sigma+\frac{3}{2})_j(m+\sigma+\frac{3}{2})_j} .$$

Let us transform Z to the power form in x^2

$$Z = 2x \frac{\Gamma(\sigma+1)\Gamma(n+m+1)\Gamma(n+m+\sigma+2)}{\Gamma(m+1)\Gamma(n+\frac{1}{2})\Gamma(n+\sigma+\frac{3}{2})\Gamma(m+\sigma+2)} \sum_{i=0}^{m} (-4x^2)^i \frac{(-m)_i(n+m+\sigma+2)_i(\frac{1}{2}-n)_i(\frac{\sigma+2}{2})_i(\frac{\sigma+3}{2})_i}{(-n-m)_i(m+\sigma+2)_i(n+\sigma+\frac{3}{2})_i\, i!\,(3/2)_i} W,$$

where

$$W = \sum_{j=0}^{m-i} \frac{(2i+\sigma+1)_j(-m+i)_j(n+m+i+\sigma+2)_j(\frac{1}{2}-n+i)_j(\frac{\sigma+3}{2}+i)_j}{j!\,(-n-m+i)_j(m+\sigma+2+i)_j(m+\sigma+3/2+i)_j(\frac{\sigma+1}{2}+i)_j} .$$

The sum W is a hypergeometric function type ${}_5F_4$ which one can sum by theorem of Dougall [4]

$$W=(-1)^m\frac{\Gamma(m+\sigma+2+i)\Gamma(-\frac{1}{2}-i)\Gamma(n+\sigma+\frac{3}{2}+i)n!\,(-n-m)_i}{\Gamma(2i+\sigma+2)\Gamma(-\frac{1}{2}-m)\Gamma(n+m+\sigma+\frac{3}{2})(n+m)!}.$$

By elementary calculations we get the required result

$$\mathcal{L}=\frac{4}{\sqrt{\pi}}\times\frac{n!}{m!}\frac{\Gamma(m+3/2)\Gamma(n+m+\sigma+2)}{\Gamma(n+1/2)\Gamma(n+m+\sigma+3/2)}\sum_{i=0}^{m}\frac{(-m)_i(-n+\frac{1}{2})_i(n+m+\sigma+2)_i}{i!\,(3/2)_i(3/2)_i}x^{2i}.$$

The proof of the form of Gibbs constants for the cases $m=0$ and $n=m$ one can be obtained by methods of elementary analysis. Here we omit the details.

References

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[3] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Higher Transcendental Functions. V.1. McGraw-Hill, New York, 1953.

[4] Dougall, J.: Proc. Edinburgh Math. Soc. V.25, p.114, 19o7.

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