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ANALYTIC SEMI-GROUPS OF GROWTH $\delta \geq 0$ AND INTEPOLATION

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The spaces which are considered in the present work are intermediate between the Banach space E and the domain of definition D(A) of an unbounded linear operator A acting in E. The operator A is supposed to be without regular points, but to have N-resolvent. The notion of N-resolvent was considered in [1] , [2] , [3] . N-resolvent we shall call (following Zafievsky) the analytic operator-function $S_N(\lambda)$ defined on some domain Ω of complex plane if for $\lambda \in \Omega$

$$S_N(\lambda)Ax = AS_N(\lambda)x , \quad x \in D(A)$$

$$S_N(\lambda)(\lambda I-A)^{N+1}x = x , \quad x \in D(A^{N+1})$$

and if $S_N(\lambda)x = 0$ for all λ ,then x=0. If the operator A has (together with a N-resolvent) an ordinary resolvent , then $S_N(\lambda) = R^{N+1}(\lambda)$. But as Da Prato noted resolvent of order N (N-resolvent) can exists also in the case when the spectrum of operator A is the whole complex plane.

Let A be a closed linear operator , D(A) be dense in E and A has a N-resolvent $S_N(\lambda)$ in the halfplane $\mathrm{Re}\,\lambda > 0$. We require the operators $S_N(\lambda)$ to be bounded and their values to be elements of $D(A^{N+1})$. This is fulfilled if N=0 and if $R(\lambda)$ exists. If $N > 0$ that holds for example if A is the infinitesimal generator of an analytic semi-group of growth $N+\alpha$ ($0 \leq \alpha < 1$) or if A is a N-regular operator (this means that there exists and is bounded the operator $[(\lambda I-A)^{N+1}]^{-1}$).

It is easy to see that the operators $A^k S_N(\lambda)$ are bounded . (k=0,N+2). For every integer k : $0 \leq k \leq N$ we introduce the conditions H_k:

$$\|A^k S_N^{\prime}(\lambda)\| \leq C/\lambda^2 , \; \|A^{k+1} S_N^{\prime}(\lambda)\| \leq C/\lambda , \; \|A^{k+2} S_N^{\prime}(\lambda)\| \leq C .$$

There are examples of operators satisfying the conditions H_k ($k=\overline{0,N}$) . In the case if N=0 the existence of resolvent and the condition of positivity of operator $-A$ imply the condition H_o . Let now $N > 0$, $0 \leq k \leq N$, $\varepsilon \in (0,N-k]$ and $T(t)$ is an analytic semi-group of negative type and growth $\gamma = N-k-\varepsilon$ (this means , that $\|T(z)\| \leq C/|z|^{\gamma}$ in a sector $|\arg z| < \varphi$ ($\varphi < \pi/2$)). Let A be the infinitesimal generator of $T(t)$. Then the condition H_k is fulfilled . The condition H_N holds if k=N and $\|T(t)\| \leq C$.

The graph-norm is introduced on D(A) as follows

$$\|a\|_{D(A)} = \|a\|_E + \|Aa\|_E .$$

Let $p \geq 1$, $0 < \theta < 1$, we denote $R_k(\theta,p) = \{ a \in E: \|a\|_{R_k(\theta,p)} =$

$$\|a\|_E + \max[\|\lambda^{1+\theta} A^{k+1} S_N^{\prime}(\lambda)a\|_{L_p^*(E)}, \; \|\lambda^{\theta} A^{k+2} S_N^{\prime}(\lambda)a\|_{L_p^*(E)}] < \infty \} .$$

<u>Theorem 1.</u> There is an imbedding $D_A(\theta,p) \subset R_k(\theta,p)$ by the condition H_k.

(Here $D_A(\theta,p) = (E,D(A))_{\theta,p}$ is the interpolation space which is constructed by real method.)

Besids condition H_k we want further A^{k+1} to be a closed operator and to exists $\lim_{\lambda \to 0} A^{k+1} S_N(\lambda)a = a_1$.

<u>Theorem 2.</u> If $a \in R_k(\theta,p)$, then $a_1 \in D_A(\theta,p)$.

Remark 1. If a bounded operator $S_N(0)$ exists , then $a_1 = A^{k+1} S_N(0)a$.

Remark 2. If we take A from the example about the fulfilment of H_k then $(A^{N-k})^{-1} = S_{N-k-1}(0)$ exists and the theorem 2 says that if a $R_k(\theta,p)$, then $(A^{N-k})^{-1} a \in D_A(\theta,p)$.

As a conseqence from theorems 1 and 2 we obtain that if k=N $D_A(\theta,p) = R_N(\theta,p)$, hence a known result follows if $R(\lambda)$ exists , namely $(E,D(A^m))_{\theta,p} = \{ a \in E: \lambda^{\theta m} [AR(\lambda)]^m a \in L_p^*(E) \}$.

Further we consider a connection between the theory of interpolation and the theory of analytic semi-groups of growth $\gamma \geq 0$. $S_N(\lambda)$, where $N = [\gamma]$, exists when A is infinitesimal generator of such semi-group.

Lets introduce the spaces ($\alpha = \gamma - N$, $0 < \theta < 1, p \geq 1, \delta > 0$)

$$T_A(\alpha,\theta,p)=\{a\in E: \|a\|_T=\|a\|_E + \|t^{N+\alpha-\theta}(T(t)a-a)\|_{L_p^*((0,\delta),E)}<\infty\}$$

and

$$S_A(\alpha,\theta,p)=\{a\in E: \|a\|_S=\|a\|_E + \|\lambda^{\theta-\alpha-N}(\lambda^{N+1}S_N(\lambda)a-a)\|_{L_p(\delta^{-1},\infty),E)}<\infty\} \ .$$

We remark that if $N=0$, $\delta=\infty$, the space $S_A(\alpha,\theta,p)$ coincides with the space $R_A(\alpha,\theta,p)$ which was introduced in [4] .

The imbeddings of $(E,D(A^{N+1}))_{\theta,p}$ in $T_A(\alpha,\theta,p)$ for $\delta<\infty$ and $T_A(\alpha,\theta,p)$ in $S_A(\alpha,\theta,p)$ were mentioned in [5] .

Lets denote by $H_{A,k}(\alpha,\theta,p)$, where $1\le k\le N+1$ the space of those $a\in E$ for which

$t^{N+\alpha+(1-\theta)k}\|A^kT(t)a\|_E \in L_p^*$ with the norm

$\|a\|_H=\|a\|_E + \|t^{N+\alpha+(1-\theta)k}A^kT(t)a\|_{L_p^*(E)}$

and by $P_{A,k}(\alpha,\theta,p)$ the space of those $a\in E$, for which is fulfilled the condition $\lambda^{k\theta-\alpha}\|A^kS_N^{(k-1)}(\lambda)\ a\|_E\in L_p^*$ with respective norm .

<u>Theorem 3.</u> There exists the imbedding $(E,D(A^k))_{\theta,p}$ in $H_{A,k}(\alpha,\theta,p)$. If $k\theta>\alpha$, then $H_{A,k}(\alpha,\theta,p)\subset P_{A,k}(\alpha,\theta,p)$ holds .

If $N=0$ from this theorm we can obtain the result from [4] . We remark that the case $k=1$ was considered by us in [5] and there was indicated that it can be generalized .

<u>Proposition .</u> If a $H_{A,k}(\alpha,\theta,p)$, then $t^{N+\alpha-k\theta+1/p'}T(t)a\xrightarrow[t\to 0]{} 0$ (Here $1\le k\le N$, $1/p+1/p'=1$). If $k=N+1$ then the statement holds by the additional condition $\theta<(N+\alpha+1/p')/(N+1)$.

<u>Corollary .</u> Let $a\in(E,D(A^k))_{\theta,p}$ and ε is an arbitrary positive number .Then we have for $1\le k\le N$ that $t^{N+\alpha+\varepsilon-k\theta}T(t)a\xrightarrow[t\to 0]{} 0$. If $k=N+1$ the statement holds by the additional condition $(N+\alpha)/(N+1)\geqslant\theta$.

In [3] there are considered semi-groups of class $\mathbb{L}^{(N)}$,containing analytic semi-groups of growth $N+\alpha$ .There was obtained that $t^{N+1-k}T(t)a\xrightarrow[t\to 0]{} 0$ if $a\in D(A^k)$, $1\le k\le N$ and that $(T(t)a-a)t^0\xrightarrow[t\to 0]{} 0$ if $a\in D(A^{N+1})$. As a corollary of the above results a decreasing of the power of t by $(1-\alpha-\varepsilon)$ for analytic semi-groups was obtained .

Let us consider the abstract Cauchy - problem $Au(t)=u'(t)$, $u(0)=u_0$. We obtain a simple corollary when A is infinitesimal generator of ordinary analytic semi-group $T(t)$ ($\gamma=N+\alpha=0$) . If $u_0\in D_A(\theta,p)$ then about the derivatives of solution we have

$$t^{m-\theta}\|u^{(m)}(t)\| \leqslant C \qquad (m \geqslant 1).$$

If T(t) is analytic semi-group of growth $\gamma > 0$ there are other statements about the Cauchy - problem.

References

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