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SOME INEQUALITIES IN WEIGHTED SOBOLEV SPACES

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0. Introduction. The aim of this paper is to find conditions on the weight functions $a_0, a_1, \dots, a_N$ which guarantee the validity of the estimate

(0.1) $$\int_\Omega |u(x)|^p a_0(x)\,dx \leq c \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p a_i(x)\,dx$$

for a rather wide class of functions $u = u(x)$ with a constant $c > 0$ independent of u. Here Ω is a set in $\mathbb{R}^N$, p is a real number, $1 < p < \infty$, and $a_i = a_i(x)$ $(i = 0,1,\dots,N)$ are measurable, almost everywhere in Ω positive functions.

For the case $N = 1$, $\Omega = (\alpha,\beta)$, inequality (0.1) has the form

(0.2) $$\int_\alpha^\beta |u(t)|^p a_0(t)\,dt \leq c\int_\alpha^\beta |u'(t)|^p a_1(t)\,dt\,;$$

this is the *generalized Hardy inequality* and therefore, (0.1) can be treated as an N-*dimensional Hardy inequality*. A special case of (0.2) is the *classical Hardy inequality*

(0.3) $$\int_0^\infty |u(t)|^p t^{\varepsilon-p}\,dt \leq \left(\frac{p}{|\varepsilon - p + 1|}\right)^p \int_0^\infty |u'(t)|^p t^{\varepsilon}\,dt$$

which holds with $\varepsilon \neq p - 1$ for a certain class of functions u, e.g. for $u \in C_0^\infty(0,\infty)$.

Necessary and/or sufficient conditions on the weight functions a_0, a_1 which guarantee the validity of (0.3), have been derived by various authors (see, e.g., B. MUCKENHOUPT [5] or A. KUFNER [2]). P.R. BEESACK [1] expressed this condition in terms of solvability of a certain differential equation in which the weight functions appear as coefficients; here we shall extend his idea to the N-dimensional case.

Inequalities of the type (0.1) have been investigated for the case $p = 2$ e.g. by R. T. LEWIS [4] and by the authors [3]; in [4], the case $a_0(x) = |\Delta g(x)|$, $a_i(x) = |\nabla g(x)|^2/|\Delta g(x)|$ $(i = 1,\dots,N)$, $c = 4$, is treated, with g a sufficiently smooth function. The case $a_i(x) = [\operatorname{dist}(x,M)]^\varepsilon$ $(i = 1,\dots,N)$ and $a_0(x) = [\operatorname{dist}(x,M)]^{\varepsilon-p}$ with $\varepsilon \in \mathbb{R}$ and M a part of the boundary $\partial\Omega$ of Ω is dealt with in [2]; in particular,

$$(0.4)\qquad \int_\Omega |u(x)|^p\, |x - x_0|^{\varepsilon-p}\, dx \le \left(\frac{p}{|\varepsilon - p + N|}\right)^p \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p |x - x_0|^\varepsilon\, dx$$

holds with $x_0 \in \partial\Omega$ and $\varepsilon \ne p - N$ e.g. for $u \in C_0^\infty(\Omega)$. Also more general weights of the type $a_i(x) = \phi_i(\operatorname{dist}(x,M))$ with $\phi_i = \phi_i(t)$ are considered in [2]. Inequality (0.4) is an obvious generalization of (0.3); we shall show here that (0.4) takes place for $x_0 \in \Omega$, too, naturally with some changes in the admissible set of functions u (see Example 3).

On the 1981 Conference on Constructive Function Theory, the authors (see [3]) presented among other inequality (0.1) for $p = 2$ and announced an analogous result for general $p > 1$. The following theorem generalizes this announced result.

1. Theorem. *Let* $p > 1$, $\Omega \subset \mathbb{R}^N$, $\Omega = \lim_{n\to\infty} \Omega_n$ *with* $\Omega_n \subset \Omega_{n+1} \subset \Omega$, $\Omega_n \in C^{0,1}$ (i.e., Ω_n *are domains in* $\mathbb{R}^N$ *with locally Lipschitzian boundaries* $\partial\Omega_n$). *Let the weight functions* $a_0, a_1, \dots, a_N$ *and the functions* u *and* v *fulfil the following conditions:*

(i) *The derivatives* $\partial u/\partial x_i$ $(i = 1,\dots,N)$ *exist a.e. in* Ω.

(ii) *The derivatives*

$$\frac{\partial a_i}{\partial x_i},\ \frac{\partial}{\partial x_i}\left[a_i\left|\frac{\partial v}{\partial x_i}\right|^{p-1} \operatorname{sgn}\frac{\partial v}{\partial x_i}\right],\quad i = 1,\dots,N,$$

exist a.e. in Ω.

(iii) *The function* v *is a solution of the (partial) differential equation*

$$(1.1)\qquad \sum_{i=1}^N \frac{\partial}{\partial x_i}\left[a_i(x)\left|\frac{\partial v}{\partial x_i}\right|^{p-1} \operatorname{sgn}\frac{\partial v}{\partial x_i}\right] + a_0(x)|v|^{p-1} \operatorname{sgn} v = 0$$

and $v \ne 0$, $\partial v/\partial x_i \ne 0$ *a.e. in* Ω $(i = 1,\dots,N)$.

(iv) *If we denote*

$$w_i = \frac{\left|\frac{\partial v}{\partial x_i}\right|^{p-1} \operatorname{sgn} \frac{\partial v}{\partial x_i}}{|v|^{p-1} \operatorname{sgn} v} , \quad i = 1,\dots,N ,$$

then

$$(1.2) \qquad |u|^p \sum_{i=1}^{N} w_i a_i \in W^{1,1}(\Omega_n) \quad \textit{for all} \quad n \in \mathbb{N} .$$

(v) *Further,*

$$(1.3) \qquad \limsup_{n\to\infty} \int_{\partial\Omega_n} |u|^p \sum_{i=1}^{N} w_i a_i \nu_{ni} \, dS \geq 0$$

where ν_{ni} *are the components of the vector* ν_n *of the outer normal to* $\partial\Omega_n$.

Then inequality (0.1) *holds with* $c = 1$.

2. Remarks. (i) If Ω is a domain in $\mathbb{R}^N$, then the domains $\Omega_n \in C^{0,1}$ obviously exist. If $\Omega \in C^{0,1}$, then it is possible to take $\Omega_n = \Omega$, but in some cases this choice is not the best one.

(ii) Roughly speaking, inequality (0.1) takes place if there exists a solution v of the (generally non-linear) partial differential equation (1.1).

(iii) If we take $\Omega \in C^{0,1}$, $\Omega_n = \Omega$, then inequality (0.1) holds for all $u \in C_0^\infty(\Omega)$, since condition (1.3) of Theorem 1 is fulfilled (provided equation (1.1) has a solution - see part (ii) of this Remark - and condition (1.2) is satisfied).

(iv) Let us take $\Omega \in C^{0,1}$, $\Omega_n = \Omega$ and let the solution v of (1.1) fulfil the non-linear boundary condition

$$(2.1) \qquad \sum_{i=1}^{N} \left|\frac{\partial v}{\partial x_i}\right|^{p-1} \operatorname{sgn} \frac{\partial v}{\partial x_i} a_i \nu_i = 0 \quad \text{on} \quad \partial\Omega$$

($\nu = (\nu_1,\dots,\nu_N)$ is the outer normal to $\partial\Omega$). In this case, inequality (0.1) holds for all $u \in C^1(\overline{\Omega})$ provided the solution v of the non-linear Neumann problem (1.1), (2.1) satisfies condition (1.2).

3. Example. Let $\Omega \in C^{0,1}$, $p > 1$, $x_0 \in \overline{\Omega}$. Then inequality

$$(3.1) \qquad \int_\Omega |u(x)|^p \, |x - x_0|^{\varepsilon-p} \, dx \leq$$

$$\leq \left(\frac{p}{|\varepsilon - p + N|}\right)^p \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p |x - x_0|^\varepsilon \left(\frac{|x_i - x_{0i}|}{|x - x_0|}\right)^{2-p} dx$$

holds if

(i) $\varepsilon < p - N$ and $u \in W^{1,1}(\Omega)$ such that $\operatorname{supp} u \cap \{x_0\} = \emptyset$, $u = 0$ on $\partial\Omega_-(x_0)$, or

(ii) $\varepsilon > p - N$ and $u \in W^{1,1}(\Omega_n)$ such that $u = 0$ on $\partial\Omega^+(x_0)$.

Here $\partial\Omega^{+(-)}(x_0) = \{x \in \partial\Omega, \sum_{i=1}^{N}(x_i - x_{0i})\nu_i > 0\ (< 0)\}$. In this case, we take $\Omega_n = \Omega - \{x \in \mathbb{R}^N, |x - x_0| < \frac{1}{n}\}$ and it can be shown that condition (1.3) is satisfied. The solution v of the corresponding equation (1.1) has the form $v(x) = |x - x_0|^{\alpha}$ with $\alpha = 1 - (\varepsilon + N)/p$.

If $1 < p < 2$, then $\left(\frac{|x_i - x_{0i}|}{|x - x_0|}\right)^{2-p} \leq 1$ and inequality (3.1) implies inequality (0.4).

4. Example. Let Ω be an unbounded domain in $\mathbb{R}^N$ such that $\Omega_n = \Omega \cap \{x \in \mathbb{R}^N, |x - x_0| < n\} \in C^{0,1}$. Further, let $p > 1$ and $x_0 \notin \bar{\Omega}$. Then inequality

$$\int_\Omega |u(x)|^p e^{\beta|x-x_0|}|x - x_0|^{\gamma-p+1}\,dx \leq \tag{4.1}$$

$$\leq \left|\frac{p-1}{\beta}\right|^{p-1}(\gamma - p + 1 + N)\cdot$$

$$\cdot \operatorname{sgn}\beta \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p e^{\beta|x-x_0|}|x - x_0|^{\gamma}|x_i - x_{0i}|^{2-p}\,dx$$

holds if

(i) $\beta > 0$, $\gamma > p - N - 1$ and $u \in W^{1,1}(\Omega)$ such that $\operatorname{supp} u$ is compact in $\mathbb{R}^N$, $u = 0$ on $\partial\Omega^+(x_0)$, or

(ii) $\beta < 0$, $\gamma < p - N - 1$ and $u \in W^{1,1}(\Omega_n)$ for every n, $u = 0$ on $\partial\Omega^-(x_0)$.

If $x_0 \in \bar{\Omega}$, then we take $\Omega_n = \left[\Omega - \{x \in \mathbb{R}^n, |x - x_0| < \frac{1}{n}\}\right] \cap \{x \in \mathbb{R}^n, |x - x_0| < n\}$, and inequality (4.1) holds if we suppose in addition in part (ii) that $\operatorname{supp} u \cap \{x_0\} = \emptyset$.

The solution v of the corresponding equation (1.1) has the form $v(x) = e^{\alpha|x-x_0|}$ with $\alpha = \beta/(1 - p)$.

5. Remark. The proof of Theorem 1 as well as further examples can be found in [6].

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