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ON THE DYADIC HAAR DERIVATIVE

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In the paper [2] W. Splettstösser and H.J. Wagner introduced the concept of dyadic Haar derivative. They proved that the Haar functions are eigenfunctions of this derivative and investigated the inverse of the derivative (the so-called dyadic Haar integral). In this paper we are concerned with the almost everywhere dyadic Haar differentiability of dyadic Haar integrals of Lebesgue integrable functions on $[0,1)$.

Let $\{\chi_n : n\in \mathbb{N}\}$ be the Haar system on $[0,1]$ and denote by $\hat{f}(n)$ $(n\in \mathbb{N})$ the n^{th} Haar-Fourier coefficient of the function $f\in L^1(0,1)$. If $h\in[0,1)$, we define the Haar translation f_h of f in the following way:

$$\hat{f}_h(0) := 0$$

$$\hat{f}_h(2^k+j) := \frac{1}{\sqrt{2^k}} \chi_{2^k}(h)\, \hat{f}(2^k+j) \qquad (k,j\in \mathbb{N},\quad 0\le j<2^k)$$

Using this notion of Haar translation we can define the dyadic Haar derivative. Let $f\in L^1(0,1)$, $n\in \mathbb{N}$ and

$$d_n f := \frac{1}{3} \sum_{j=0}^{n-1} 2^j (f - f_{2^{-j-1}}) +$$

$$+ \frac{1}{3} (f - \int_0^1 f)$$

the n^{th} dyadic "difference quotient" of the function f. We say that f is dyadic Haar differentiable in the strong sense if there exists a function $Df \in L^1(0,1)$ (the strong dyadic derivative of f) for which

$$\lim_n \| d_n f - Df \|_1 = 0$$

($\| \cdot \|_1$ denotes the L^1-norm). Analogously, we say that f is dyadic Haar differentiable at the point $x \in [0,1)$ if

$$\lim_n (d_n f)(x)$$

exists. This is denoted by $f^{[1]}(x)$ and called the pointwise dyadic derivative of f at x.

Let

$$K(x) := 1 + \sum_{n=0}^{\infty} 2^{-n/2} \chi_{2^n}(x) \quad (x \in [0,1)) .$$

It is obvious that $K \in L^1(0,1)$ and $\| K \|_1 = 1$. If $f \in L^1(0,1)$ we define the dyadic Haar integral of f by the following equality:

$$If := f * K ,$$

where $*$ denotes the dyadic convolution on $[0,1)$. It was proved in [2] that the integral If is dyadic Haar differentiable and

$$D(If) = f \qquad (f\in L^1(0,1), \int_0^1 f = 0) .$$

In this paper we prove the pointwise analogue of this statement. This is a consequence of a maximal theorem.

Let T denote the maximal operator of the sequence $d_n \circ I$ $(n\in \mathbb{N})$, i.e. let

$$(Tf)(x) := \sup_{n\in\mathbb{N}} |d_n(If)(x)| \qquad (f\in L^1(0,1), \; x\in[0,1)) .$$

For T we have the following

<u>THEOREM.</u> The operator T is of type (∞, ∞) and of weak type $(1,1)$, i.e., there exist constants $C_1, C_2 > 0$ such that

$$\|Tf\|_\infty \leq C_1 \|f\|_\infty \qquad (f\in L^\infty(0,1))$$

and

$$\text{mes}\{x\in[0,1): (Tf)(x)>\lambda\} \leq C_2 \frac{\|f\|_1}{\lambda} \qquad \left(f\in L^1(0,1), \; \lambda > 0 \right) .$$

From this theorem follows the following

<u>COROLLARY.</u> If $f\in L^1(0,1)$ and $\int_0^1 f = 0$, then we have

$$(If)^{[1]}(x) = f(x) \qquad (\text{a.e. } x\in[0,1)) .$$

<u>Proof of the Theorem.</u> It is easy to see that

$$d_n(If)(x) = \frac{2^n}{3}[(If)(x) - S_{2^n}(If)(x)] +$$

$$+ S_{2^n}(f)(x)$$

$$(f\in L^1(0,1), \quad x\in[0,1), \quad n\in \mathbb{N}) ,$$

where S_{2^n} denotes the 2^n th partial sum operator with respect to the Haar system (see [2], p. 534). Since the maximal operator of the sequence S_{2^n} $(n\in \mathbb{N})$ is of type (∞, ∞) and of weak type $(1,1)$, we have to work with the first term only. This we may write in the following form:

$$\frac{2^n}{3}[(If)(x) - S_{2^n}(If)(x)] =$$

$$= \frac{2^n}{3}[(K - S_{2^n}(K)) * f](x) = \frac{2^n}{3}(K^{(n)} * f)(x) ,$$

where

$$K^{(n)}(x) := \sum_{j=n}^{\infty} 2^{-j/2} \cdot \chi_{2^j}(x) \quad (x\in[0,1], \quad n\in \mathbb{N}) .$$

Let Q denote the following maximal operator:

$$(Qf)(x) := \sup_{n\in \mathbb{N}} |(2^n K^{(n)} * f)(x)|$$

$$(f\in L^1(0,1), \quad x\in[0,1)) .$$

Q is of type (∞, ∞), since

$$\| 2^n K^{(n)} * f \|_\infty \leq 2^n \| K^{(n)} \|_1 \| f \|_\infty =$$

$$= \| f \| \qquad (f \in L^\infty(0,1), \quad n \in \mathbb{N})$$

Here we used the equality $\| K^{(n)} \|_1 = 2^{-n}$ $(n \in \mathbb{N})$ (see [2], p. 532).

In order to prove the weak type (1,1) property of Q, we rewrite $2^n \cdot K^{(n)}$ with the help of the equality

$$\chi_{2^j} = \frac{1}{\sqrt{2^j}} (D_{2^{j+1}} - D_{2^j}) \qquad (j \in \mathbb{N})$$

as follows:

$$2^n K^{(n)} = \sum_{j=n}^{\infty} \frac{1}{2^{j-n}} (D_{2^{j+1}} - D_{2^j}) =$$

$$= \sum_{j=n+1}^{\infty} \frac{1}{2^{j-n}} D_{2^j} - D_{2^n} \qquad (n \in \mathbb{N}).$$

From this form of $2^n K^{(n)}$ may be derived that the operator Q is of weak type (1,1) if we use e.g. the Theorem in [1] proved by F. Schipp.

References

[1] Schipp F., Über gewisse Maximaloperatoren, Annales Univ. Sci. Bp., Sect. Math., 18 (1975), 189-195.

[2] Splettstösser W., Wagner H.J., Eine dyadische Infinitesimalrechnung für Haar-Funktionen, Z. Angew. Math. Mech., 57 (1977), 527-541.

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