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ON ASPLUNDS'S AVERAGING METHOD - THE INTERPOLATION (FUNCTION) WAY

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0. Introduction. As is well-known, Edgar Asplund [1] invented his averaging method in order to construct new norms with better "smoothness" properties out of two given norms in one and the same Banach space. In this note we wish in the first place to interpret it as an interpolation process. More precisely, in our case the two initial norms are taken from two different spaces (forming a compatible pair of Banach spaces; cf. e.g. [2]) and the averaging process of Asplund amounts essentially to a repeated formation of K- and J-functionals. For interpolation of weighted L^2 it also amounts to a successive formation of harmonic and arithmetic means of the weight functions involved. In this way one gets a new proof of the simplest and perhaps most famous and likewise most useful of all interpolation theorems, the one of Lions [7] (1958). This invites one also to make a comparison with the classical algoritm for the geometric-arithmetic mean (associated with the name of Gauss), as well as other related processes (Schwab-Borchardt, Carlson etc.; see e.g. [2], chap. 1, [9], chap. 12). Many interesting questions of honest oldfashioned analysis arise in this way, not all of which I have yet been able to solve.

Notation. The norm in a Banach space X will in general be denoted by $||\cdot||_X$ or, by typographical reasons (if there sit to many indices on X), alternatively by $||\cdot||X$.

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1. The interpolation spaces of Asplund. Let $\bar{A} = (A_0, A_1)$ be a (compatible) pair of Banach spaces (i.e. A_0 and A_1 are both continuously imbedded in some Haudorff topological vector space $\underline{A}$). We associate with $\bar{A}$ the new pair $\bar{A}^{(1)} = (A_0^{(0)}, A_1^{(1)})$ defined via the prescription that

$$||a||_{A_0^{(1)}} = \inf_{a=a_0+a_1} 2^{1-1/p}(||a_0||^p_{A_0} + ||a_1||^p_{A_1}) \quad (= 2^{1-1/p}K_p(1,a)),$$

$$||a||_{A_1^{(1)}} = 2^{-1/p}(||a||^p_{A_0} + ||a||^p_{A_1})^{1/p} \quad (= 2^{-1/p}J_p(1,a)).$$

Here p is a fixed number, $1 < p < \infty$. Clearly $A_0^{(1)} = \Sigma = A_0 + A_1$ and $A_1^{(1)} = \Delta = A_0 \cap A_1$ topologically. Repetition of this procedure yields a sequence of pairs $\bar{A}^{(0)}$ $(\equiv \bar{A})$, $\bar{A}^{(1)}$, $\bar{A}^{(2)}, \ldots$ $(= (\Sigma,\Delta)$ topologically).

The estimates in the following two lemmata are (if $p = 2$) in essence contained in [1].

LEMMA 1. Assume that $A_0 = A_1$ with $||a||A_1 \geq ||a||A_0$. Then

$$(1)\qquad ||a||_{A_0} \leq ||a||_{A_0^{(1)}} \leq ||a||_{A_1^{(1)}} \leq ||a||_{A_1} .$$

LEMMA 2. Assume that $A_0 = A_1$ and that $||a||A_0 \leq ||a||A_1 \leq \Gamma||a||A_0$ where $\Gamma \geq 1$. Then holds (besides (1))

$$(2)\qquad ||a||_{A_1^{(1)}} \leq \Gamma_1 ||a||_{A_0^{(1)}}$$

with

$$(3)\qquad \Gamma_1 = 2^{-1}(2^{p/q}+(1+\Gamma^q)^{p/q})^{1/p} \quad (\leq \Gamma)$$

where q is the conjugate exponent, $q = p/(p-1)$ (so that $1/p+1/q = 1$).

By iteration one gets from (1) (in the hypothesis $A_0 \supset A_1$)

$$(4)\qquad ||a||_{A_0^{(0)}} \leq ||a||_{A_0^{(1)}} \leq ||a||_{A_0^{(2)}} \leq \ldots \leq ||a||_{A_1^{(2)}} \leq ||a||_{A_1^{(1)}} \leq ||a||_{A_1^{(0)}}$$

and from (3) (in the hypothesis $A_0 = A_1$)

$$(5)\qquad ||a||_{A_0^{(n)}} \leq \Gamma_n ||a||_{A_1^{(n)}}$$

with Γ_n given by

$$(6)\qquad \Gamma_n = 2^{-1}(2^{p/q}+(1+\Gamma_{n-1}^q)^{p/q})^{1/p} \quad \text{if } n \geq 1,\ \Gamma_0 = \Gamma.$$

It follows from (4) that the limits $||a||A_0^{(\infty)} = \lim_{n\to\infty}||a||A_0^{(n)}$ and $||a||A_1^{(\infty)} = \lim_{n\to\infty}||a||A_1^{(n)}$ exist. The corresponding Banach spaces $A_0^{(\infty)}$ and $A_1^{(\infty)}$ will be termed the <u>Asplund interpolation spaces</u>. Clearly $A_0^{(\infty)} \supset A_1^{(\infty)}$ and $||a||A_0^{(\infty)} \leq ||a||A_1^{(\infty)}$ (in the hypothesis $A_0 \supset A_1$). Since $\Gamma_{n+1} \leq \Gamma_n$ and $\lim_{n\to\infty}\Gamma_n = 1$ similarly $A_0^{(\infty)} = A_1^{(\infty)}$, $||a||A_0^{(\infty)} = ||a||A_1^{(\infty)}$ (in the hypothesis $A_0 = A_1$).

<u>2. The case of a Hilbert couple</u>. Let now $\bar{A} = (A_0,A_1)$ be a Hilbert couple. By virtue of the spectral theorem it is no loss of generality to assume that we have to deal with direct integrals of Hilbert spaces over the interval $(0,\infty)$, and for simplicity let us only consider the case of spectral multiplicity one. Thus we assume that

$$(1)\qquad ||a||_{A_0} = [\int_0^\infty |a(x)|^2 d\mu(x)]^{1/2}, \qquad ||a||_{A_1} = [\int_0^\infty |a(x)|^2 d\mu(x)]^{1/2} ,$$

μ being a positive measure on $(0,\infty)$. Thus $A_0 = L^2(\mu)$, $A_1 = L^2(x,\mu)$ (weighted L^2). We adjust the previous parameter p to the situation in hand, thus taking $p = q = 2$.

We then find easily (cf. 8)

$$||a||_{A_n^{(1)}} = [\int_0^\infty |(\frac{2}{1+\frac{1}{x^2}})^{1/2} a(x)|^2 d\mu(x)]^{1/2}, \quad ||a||_{A_n^{(1)}} = [\int_0^\infty |(\frac{1+x^2}{2})^{1/2} a(x)|^2 d\mu(x)]^{1/2}.$$

Hence by iteration

$$||a||_{A_0^{(n)}} = [\int_0^\infty |H_{0n}(x) a(x)|^2 d\mu(x)]^{1/2}, \quad ||a||_{A_1^{(n)}} = [\int_0^\infty |H_{1n}(x) a(x)|^2 d\mu(x)]^{1/2}.$$

with $H_0^{(n)}$ and $H_1^{(n)}$ giben by

$$(2) \qquad (H_0^{(n)})^2 = \frac{2}{\frac{1}{(H_0^{(n-1)})^2} + \frac{1}{(H_1^{(n-1)})^2}} \quad \text{(harmonic mean)}$$

$$(H_1^{(n)})^2 = \frac{(H_0^{(n-1)})^2 + (H_1^{(n-1)})^2}{2} \quad \text{(arithmetic mean)}$$

with $H_0^{(0)} \equiv 1$, $H_1^{(0)} \equiv x$. Notice that $H_0^{(n)} H_1^{(n)} = H_0^{(n-1)} H_1^{(n-1)} = \ldots = x$. Therefore $\lim_{n\to\infty} H_0^{(n)} = \lim_{n\to\infty} H_1^{(n)} = \sqrt{x}$ (geometric mean). Thus in this case the two Asplund spaces coincide, $A_0^{(\infty)} = A_1^{(\infty)} = L^2(\sqrt{x},\mu)$.

At the same time we have proved that $H(x) = \sqrt{x}$ is an interpolation function (definition in sec. 3). This implies that $H(x) = x^\theta$ too is an interpolation function with $0 \leq \theta \leq 1$. Therefore we have here a new proof of Lions's theorem [7].

If we specialize the previous considerations to the case of scalar couples $\bar{A} = (A_0, A_1)$ we obtain the following consequence: The Asplund methods have "characteristic function" $\sqrt{x}$. Therefore, by abstract nonsense:

$$A_{1/2,\infty;K} \supset A_0^{(\infty)} \supset A_1^{(\infty)} \supset A_{1/2,1;J}.$$

QUESTION. Under which circumstance is it true that $A_0^{(\infty)} = A_1^{(\infty)}$?

REMARK (added Apr. 84). After the above was written, Svante Janson has kindly pointed out to me that at least the two norms trivially coincide on Δ. He has also some other results (unpublished) pertaining to this conjecture.

3. Weighted L^p. As a generalization of the situation considered in the previous § let us now give a look at the pair $(L^p(\mu), L^p(x,\mu))$. Stein and Weiss [15] (1958) proved that $L^p(x^\theta,\mu)$, where $\theta \in (0,1)$, is an interpolation space for this pair, in other words, that x^θ is an interpolation function. If $p = 2$ this is essentially just Lions's theorem 6 treated above.

Naively one could now believe that the previous procedure for $p = 2$ would extend to the present case, so as to produce a new proof of the Stein-Weiss theorem too. This is however not that easy. Let us see why. A simple analysis shows that the formulae (2) of sec. 2 have the following counter-parts:

$$(1)\quad\begin{cases} H_0^{(n)} = ((H_0^{(n-1)})^{-q} + (H_1^{(n-1)})^{-q}/2)^{-1/q} \\ H_1^{(n)} = ((H_0^{(n-1)})^{p} + (H_1^{(n-1)})^{p}/2)^{1/p} \end{cases} .$$

(harmonic and arithmetic power means)

It is easy to see that the limit of the means involved still exists. Let the limit function be denoted H^*, $H^* = \lim_{n\to\infty} H_0^{(n)} = \lim_{n\to\infty} H_1^{(n)}$. So the question is, what is this H^*? Probably H^* does not admit a representation in terms of "known" functions. However, it is easy to prove (see [12]) that the following asymptotic representation is valid:

$$H^*(x) \sim \begin{cases} Lx^{1/q} & \text{for } x \to \infty \\ Lx^{1/p} & \text{for } x \to 0 \end{cases} .$$

Here L is a certain numerical constant depending on p, $L = 1$ exactly if $p = 2$ (in which case $H^*(x) = \sqrt{x}$, as we already know). Numerical experiments reveal that $0.8348 \leq L < 1$ with a minimum at $p \doteq 6.08$, this for the range $2 < p < \infty$; the case $1 < p < 2$ is entirely symmetrical.

Back to interpolation! The above results only show that the Asplund spaces in this case coincide with $L^p(x^{1/p},\mu)$ up to equivalence of norm. This if we also assume that $\operatorname{supp}\mu \subset (1,\infty)$. If instead $\operatorname{supp}\mu \subset (0,1)$ we get similarly the space $L^p(x^{1/q},\mu)$.

QUESTION. Is it possible to prove the Stein-Weiss theorem by "iteration", that is, by somehow modifying the Asplund process?

4. General notions on interpolation functions.

DEFINITION. $H = H(x)$ $(x \in (0,\infty))$ is said to be an (exact) interpolation function (of exponent p) if for every linear operator T the conditions

$$\int_0^\infty |Tf(x)|^p d\mu(x) \leq \int_0^\infty |f(x)|^p d\mu(x), \quad \int_0^\infty |xTf(x)|^p d\mu(x) \leq \int_0^\infty |xf(x)|^p d\mu(x),$$

for some positive measure μ imply that

$$\int_0^\infty |H(x)Tf(x)|^p d\mu(x) \leq \int_0^\infty |H(x)f(x)|^p d\mu(x)$$

for the same measure . The set of all interpolation functions (of exponent p) will be denoted by $\underline{I}^p$. Clearly:

(1) $H_1, H_2 \in \underline{I}^p \Rightarrow (H_1^p + H_2^p)^{1/p} \in \underline{I}^p$.

Almost clearly:

(2) $H \in \underline{I}^p \Rightarrow \dfrac{1}{H(x^{-1})} \in \underline{I}^p (q = p')$,

because $(L^p(x,\mu))^* \approx L^q(x^{-1},\mu)$. According to Foias-Lions [5], $H \in \underline{I}^p$ if

$$(3)\quad H(x) = \left(\int_0^\infty \frac{dS(y)}{(1+(y/x)^q)^{p/q}}\right)^{1/p}$$

for some positive measure S, or (alternatively)

$$(4)\quad H(x) = \left(\int_0^\infty \frac{dT(y)}{(1+(xy)^p)^{q/p}}\right)^{-1/q}$$

for some other measure T.

For $p = 2$ these sufficient conditions are also necessary (and equivalent) [5], [3]. The proof depends on Loewner's theory of monotone operator functions (see [4]).

"CONJECTURE" (implicit in [5]). If $1 < p < 2$ is necessary and if $2 < p < \infty$ (4) is necessary.

I use Gänsefusse " " around the word conjecture, because I think that really not much evidence has been produced for raising this to the rank of a true conjecture - and according to certain authorities one should not overuse this word attaching it to any arbitrary wild guess.

Regarding weighted L^p in general there are however basically two major schools of thought. According to one school the norm should be taken to be

$$||f|| = (\int (W(x)|f(x)|^p d\mu(x))^{1/p},$$

where as according to the followers of another one should take

$$||f|| = (\int |f(x)|^p w(x) d\mu(x))^{1/p}.$$

Apparently

$$(5) \qquad w(x) = (W(x))^p, \; W(x) = (w(x))^{1/p}.$$

Corresponding to this we have really two types of interpolation functions:

1^o interpolation functions of the upper class (those which we have employed until now),

2^o those of the lower class (to which we turn hereafter).

The set of all lower class interpolation functions will be denoted by $\underline{I}_p$. Clear (see (5)): $h \in \underline{I}_p \iff H \in \underline{I}^p$ where $H(x) = (h(x^p))^{1/p}$. Translation:

$$(1') \qquad h_1, h_2 \in \underline{I}_p \Rightarrow h_1 + h_2 \in \underline{I}_p \quad (\underline{I}_p \text{ is a cone!})$$

$$(2') \qquad h \in \underline{I}_p \Rightarrow \frac{1}{h(x^{-p/q})^{q/p}} \in \underline{I}_q \qquad (\text{with } q = p').$$

$$(3') \qquad h(x) = \int_0^\infty \frac{ds(y)}{(1+(y/x)^{q/p})^{p/q}} \Rightarrow h \in \underline{I}_p.$$

$$(4') \qquad h(x) = (\int_0^\infty \frac{dt(y)}{(1+xy)^{q/p}})^{-q/p} \Rightarrow h \in \underline{I}_p.$$

Until now we have dealt with interpolation of linear operators exclusively. Let us now also invoke quasi-linear operators. By a quasi-linear operator we mean an operator T transforming measurable functions into measurable functions such that 1^o $|T(\lambda f)| = \lambda |f|$ $(\lambda > 0)$, 2^o $|T(f+g)| \leq |Tf| + |Tg|$. To this there corresponds in an obvious manner a notion of interpolation function. The set of all "quasi-linear" interpolation functions (of lower class and exponent p) shall be denoted by $\underline{I}_p$. Clear: $\underline{\tilde{I}}_p \subset \underline{\tilde{I}}_p$.

PROPOSITION. $\underline{\tilde{I}}_p \subset \underline{\tilde{I}}_r$ if $p \geq r$.

5. The geometric-arithmetic mean from the interpolation point of view. The Asplund algorithm of sec. 1-2 ("harmonic-arithmetic mean") leads one "zwangslos" to think of other famous algorithms, in the first instance the one defining the geometric-arithmetic mean usually associated with the name of Gauss. Let us briefly recall here the salient features about the latter.

Let a and b be any two positive numbers. Then we can associate with them two positive (numerical) sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_{n+1} = \sqrt{a_n b_n},\quad b_{n+1} = (a_n+b_n)/2 \quad (n > 0),\quad a_0 = a,\quad b_0 = b.$$

As is well-known both sequences converge to the same limit and this limit is by definition the geometric-arithmetic mean of a and b. It is usually denoted by $M(a,b)$. It is convenient to put $M^*(a,b) = (M(1/a,1/b))^{-1}$.

As has been retold many times (cf. e.g. [6]), Gauss as a kid, while other youngsters were doing mischief mainly, busied himself with the numerical computation of the geometric-arithmetic mean. This eventually gave a payoff and lead him to spectacular discoveries in the area of elliptic (modular) functions (integrals). Moral? In particular, he proved that a formula which for our purposes we state as

$$\pi/2\ M^*(1,x) = \frac{\pi/2}{M(1,1/x)} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{\cos^2\varphi + 1/x^2 \sin^2\varphi}}$$

The right hand side is a complete elliptic integral. Even better, let us rewrite the latter making a change of variables ($y = \tan\varphi$, $dy = d\varphi/\cos^2\varphi$):

$$\text{(1)}\qquad \pi/2\ M^*(1,x) = \int_0^\infty \frac{dy}{\sqrt{1+y^2}\cdot\sqrt{1+(y/x)^2}}$$

Notice that this a <u>convolution</u> on the multiplicative $R_+^* = (0,\infty)$.

Let us see how this relates to interpolation functions! Observe that $\sqrt{x}$ is in $\underline{I}_p$ for any p (by [15] or by [5]). Therefore if we start out with the functions 1 and x in $\underline{I}_p$ and repeatedly take the geometric and the aritmetic mean we see in the limit that $M(1,x)$ must be in $\underline{I}_p$. Let us concentrate on the case p = 2. Thus $M(1,x) \in \underline{I}_2$ $(=\tilde{\underline{I}}_2)$ and so $M^*(1,x) \in \underline{I}_2$. Thus by Loewner's theorem there must exist an integral representation of the type

$$\text{(2)}\qquad M^*(1,x) = \int_0^\infty \frac{ds(y)}{1+\frac{y}{x}}\ .$$

But this is convolution with the "wrong" kernel, $1/(1+1/x)$, not $\sqrt{1+(1/x)^2}$. So what is the interpolation interpretation of Gauss's theorem? It took me quite a time to figure out a reasonable answer:

$$M^*(1,x) \in \tilde{\underline{I}}_{3/2} \quad (\subset \underline{I}_{3/2}).$$

This is connected with the following general conjecture (but remember what we just said about conjecturing!):

CONJECTURE. $\underline{I}_p \subset \underline{I}_r$ if $p \geq r$.

6. Integral representations. Let us introduce the class $\underline{V}_a$ of all functions f admitting a representation of the type $f = k_a * ds$, s a positive measure on the interval $(0,\infty)$ and $k_a \overset{\text{def}}{=} x(1+x^a)^{1/a}$. Explicitly:

$$f(x) = \int_0^\infty \frac{1}{(1+(\frac{y}{x})^a)^{1/a}}\, ds(y).$$

Thus in the previous notation (sec. 4): $\underline{I}_p \supset \underline{\tilde{I}}_p \supset \underline{V}_a$, $a = 1/(p-1)$, and it was "conjectured" that this really is an identity. This again suggests the following new conjecture.

CONJECTURE. $\underline{V}_a \subset \underline{V}_b$ $(\subset \underline{V}_\infty)$ if $a \leq b$.

The last inclusion is certainly true, because formally $\underline{V}_\infty$ = convex functions, whereas $\underline{V}_1$ = Loewner functions. This because the kernel k_a is a concave function. However, we have only been able to verify this conjecture in a special case.

PROPOSITION. If $b = na$ (n integer > 1) then $k_b = k_a * s$ where $s = \text{const}$ $x^{1+a}/(1+x^b)^{1/b+2/n} * x^{1+2a}/(1+x^b)^{1/b+4/n} * \ldots * x^{1+(n-1)b}/(1+x^b)^{1/b+2(n-1)/n}$ (n-1 factors). (The principle of the proof (omitted) was communicated to me by Lars Vretare.)

COROLLARY. If $b = na$ (n integer > 1) then $\underline{V}_a \subset \underline{V}_b$.

Finally we discuss the problem of reconstructing the measure s from the function.

REMARK. Notice that the functions in $\underline{V}_a$ admit analytic continuation to the angle $|\arg z| < \pi/a$.

Let us mention two known special cases.

$a = \infty$. In this case $ds = -xf''\,dx$. This is really very elementary (essentially an integration by parts). The result is often used in interpolation theory (see e.g. Sparr [10]).

$a = 1$. Now $ds = 1/\pi \cdot \operatorname{Im} f(-x)$ where $f(-x)$ stands for the boundary value of the analytic continuation of f to the set $|\arg z| < \pi$. This is essentially Pick's theorem (see again [4]).

We thus wish to interconnect these two known cases. If $a < 1$ we get the following formula:

$$\boxed{s = \frac{1}{\pi}\,\frac{a^2\,\Gamma(\frac{1}{a})}{\Gamma(\frac{1}{a}-1)}\,(1-x^a)_+^{1/a-2}\,x^a * f_a\,.}$$

Here $f_a = (f_a^+ - f_a^-)/2i$ and f_a^+ and f_a^- denote the boundary values of the analytic continuation of f on the two boundary lines $\arg z = \pi/a$ and $\arg z = -\pi/a$ respectively. It is clear that this formula contains as a limiting case ($a = 1$) Pick's formula just quoted. (The convoluting measure reduces itself to a delta function!)

Similarly if $a > 1$ we have

$$\boxed{s = \frac{1}{\pi}\,\frac{a\,\Gamma(\frac{1}{a})}{\Gamma(\frac{1}{a}-1)}\,(1-x\frac{d}{dx})\{(1-x^a)_+^{1/a}\,x^a * f_a\}.}$$

Again we recognize in the limit ($a \to \infty$) Sparr's formula [10]: $s = (1-xd/dx)\{df/dx\}$.

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