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ON THE RELATIONS BETWEEN RATIONAL AND SPLINE APPROXIMATIONS

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In this paper we study some relations between the best rational and spline approximations of functions in L_p metric.

Denote by $R_n(f)_p$ the best approximation of the function f in $L_p[a,b]$, $1 \leq p \leq \infty$, by rational functions of degree at most n and by $S_n^k(f)_p$ the best approximation of f in $L_p[a,b]$ by piece-wise polinomial functions of degree $k-1$ with $n+1$ free knots on $[a,b]$.

V.A.Popov proved the following statement:

Theorem 1 [1] . If $f \in C[0,1]$, then

$$S_n^1(f)_\infty \leq 2^7 \frac{\sum_{\nu=0}^{n} R_\nu(f)_\infty}{n+1} \qquad \text{for} \quad n \geq 1 .$$

We obtained the following two results:

Theorem 2 [2] . If $1 \leq p < q \leq \infty$, $f \in L_q[a,b]$, $\alpha > 0$ and $k \geq 1$, then

$$R_n(f)_p \leq C \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} S_\nu^k(f)_q}{n^\alpha} \qquad \text{for} \quad n \geq k-1 ,$$

where $C = C(p,q,\alpha,k,a,b)$ depends only on p,q,α,k,a and b .

Theorem 3 [2] . If $1 \leq p < q \leq \infty$, $f \in L_q[a,b]$, $r \geq 1$ and $0 < \alpha < r$, then

$$S_n^r(f)_p \leq C \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f)_q}{n^\alpha} \qquad \text{for} \quad n \geq 1 , \quad ,$$

where $C = C(p, q, r, \alpha, a, b)$.

Some relations between the rational and spline approximations of functions and their derivatives are obtained in [3] and [4]. V.A.Popov [5] obtained connection between the rational uniform approximations of functions and the polynomial approximations of their derivatives in L_p metric. The results of Yu.A.Brudnyi from [6] are closely connected to our results.

The aim of this paper is to announce the following improvement of Theorem 2:

Theorem 4. If $f \in L_p[a, b]$, $1 \le p < \infty$, $k \ge 1$ and $\alpha > 0$, then

$$R_n(f)_p \le C \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} \cdot S_\nu^k(f)_p}{n^\alpha} \quad \text{for } n \ge k-1,$$

where $C = C(p, k, \alpha)$ depends on p, k and α.

Corollary 1. If $f \in L_p[a, b]$, $1 \le p < \infty$, $k \ge 1$, $\gamma > 0$ and $S_n^k(f)_p = O(n^{-\gamma})$ then $R_n(f)_p = O(n^{-\gamma})$.

Theorem 4 shows that the rational functions as an approximation tool in $L_p (1 \le p < \infty)$ are not worse than spline functions.

The proof of Theorem 4 is based on the following statement:

Theorem 5. Let $1 \le p < \infty$, $k \ge 1$, $m \ge 1$ and let φ be a piese-wise polynomial function of degree $k-1$ with $m+1$ free knots on $[a, b]$. Let $\varphi(x) = 0$ for $x \in (-\infty, \infty) \setminus [a, b]$ Then

$$R_n(\varphi, (-\infty, \infty))_p \le 2.e^{-c\sqrt{\frac{n}{m}}} \|\varphi\|_{L_p[a,b]} \quad for \quad n \ge 1,$$

where $C = C(p, k) > 0$ depends on p and k.

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